1. Operations with binary vectors and communication over Binary Symmetric Channel (BSC)

Coding Technology

Problem 1

(a) For a BSC, the input vector is $u = (0010011)$ and the randomly generated error vector is $e = (1000001)$. The bit error probability is $P_b = 0.1$. What is the output vector of the channel?

(b) What is the probability of the error vector $e = (1000001)$?

Problem 1

(a) For a BSC, the input vector is $u = (0010011)$ and the randomly generated error vector is $e = (1000001)$. The bit error probability is $P_b = 0.1$. What is the output vector of the channel?

(b) What is the probability of the error vector $e = (1000001)$?

Solution.

(a) The output is the mod 2 sum of the input and the error vector:

$$\begin{array}{rr} & 0010011 \\ + & 1000001 \\ \hline & 1010010 \end{array}$$

Problem 1

(a) For a BSC, the input vector is $u = (0010011)$ and the randomly generated error vector is $e = (1000001)$. The bit error probability is $P_b = 0.1$. What is the output vector of the channel?

(b) What is the probability of the error vector $e = (1000001)$?

Solution.

(a) The output is the mod 2 sum of the input and the error vector:

$$\begin{array}{rr} & 0010011 \\ + & 1000001 \\ \hline & 1010010 \end{array}$$

(b) The probability of the error vector is

$$P(e = 1000001) = 0.1^2 \cdot (1 - 0.1)^5 \approx 0.005905.$$

Problem 2

(a) $u = (0010011)$ is the input and $v = (1010010)$ is the corresponding output of a BSC. What is the Hamming distance between u and v?

(b) What is the error vector generated by the BSC?

Problem 2

(a) $u = (0010011)$ is the input and $v = (1010010)$ is the corresponding output of a BSC. What is the Hamming distance between u and v?

(b) What is the error vector generated by the BSC?

Solution.

(a)

$$\begin{array}{ll} \text{input:} & 0010011 \\ \text{output:} & 1010010 \\ & \downarrow \qquad \downarrow \\ & 1 \;\; + \;\; 1 = 2 \end{array}$$

Problem 2

(a) $u = (0010011)$ is the input and $v = (1010010)$ is the corresponding output of a BSC. What is the Hamming distance between u and v?

(b) What is the error vector generated by the BSC?

Solution.

(a)

$$\begin{array}{ll} \text{input:} & 0010011 \\ \text{output:} & 1010010 \\ & \downarrow \qquad \downarrow \\ & 1 \;+\; 1 = 2 \end{array}$$

(b) $v = u + e \implies e = u + v$

$$\begin{array}{ll} \text{input:} & 0010011 \\ \text{output:} & 1010010 \\ \text{error:} & 1000001 \end{array}$$

Problem 3

(a) What is the error vector if the input vector is $u = (00100111)$ output vector is $v = (10100101)$?

(b) If the channel error probability is $P_b = 0.01$, what is the conditional probability that the output vector is $v = (10100101)$, assuming that the input vector is $u = (00100111)$?

Problem 3

(a) What is the error vector if the input vector is $u = (00100111)$ output vector is $v = (10100101)$?

(b) If the channel error probability is $P_b = 0.01$, what is the conditional probability that the output vector is $v = (10100101)$, assuming that the input vector is $u = (00100111)$?

Solution.

(a) The error vector is input + output mod 2:

$$e = u + v = (10000010)$$

Problem 3

(a) What is the error vector if the input vector is $u = (00100111)$ output vector is $v = (10100101)$?

(b) If the channel error probability is $P_b = 0.01$, what is the conditional probability that the output vector is $v = (10100101)$, assuming that the input vector is $u = (00100111)$?

Solution.

(a) The error vector is input + output mod 2:

$$e = u + v = (10000010)$$

(b)

$$P(v = (10100101) | u = (00100111)) =$$
$$P_b^2(1 - P_b)^6 = 0.01^2 \cdot 0.99^6 \approx 0.00009415.$$

Problem 4

For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$.

Problem 4

For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$.

Solution. We use the following notation.

- i is the number of incorrect bits in a block;
- $30 - i$ is the number of correct bits;
- t is the number of errors we can correct.

Problem 4

For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$.

Solution. We use the following notation.

- i is the number of incorrect bits in a block;
- $30 - i$ is the number of correct bits;
- t is the number of errors we can correct.

We need to find the smallest t for which

$$\sum_{i=t+1}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \leq 0.001.$$

Problem 4

For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$.

Solution. We use the following notation.

- i is the number of incorrect bits in a block;
- $30 - i$ is the number of correct bits;
- t is the number of errors we can correct.

We need to find the smallest t for which

$$\sum_{i=t+1}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \leq 0.001.$$

$$\left.\begin{array}{l} \sum_{i=13}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \approx 0.00311 \\ \\ \sum_{i=14}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \approx 0.000902 \end{array}\right\} \implies t = 13.$$

Problem 5

(a) What is the number of 5-dimensional binary vectors with Hamming distance 3 from the vector (01010)?

(b) What is the number of binary vectors inside the sphere with radius 3 with center (01010)?

Problem 5

(a) What is the number of 5-dimensional binary vectors with Hamming distance 3 from the vector (01010)?

(b) What is the number of binary vectors inside the sphere with radius 3 with center (01010)?

Solution.

(a)

$$\binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10.$$

Problem 5

(a) What is the number of 5-dimensional binary vectors with Hamming distance 3 from the vector (01010)?

(b) What is the number of binary vectors inside the sphere with radius 3 with center (01010)?

Solution.

(a)

$$\binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10.$$

(b)

$$\sum_{i=0}^{3} \binom{5}{i} = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} =$$

$$1 + 5 + 10 + 10 = 26.$$

Problem 6

Calculate the weight of the vector (000100011000111101000).

Problem 6

Calculate the weight of the vector (0001000110001111101000).

Solution. The vector contains 8 ones, so

$$w(0001000110001111101000) = 8.$$

Problem 7

Calculate the following matrix-vector multiplication according to mod 2 arithmetics.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(Add columns 2, 3 and 7 of the matrix componentwise.)

Problem 8

Calculate the following matrix-vector multiplication according to mod 2 arithmetics.

$$(1001) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Problem 8

Solution.

$$(1001) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} = (01001110).$$

(Add rows 1 and 4 of the matrix componentwise.)