Random matrix theory with various applications
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Quantum probability and random matrices
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In the 50's Wigner, in connection with some phenomenological models of nuclear physics, investigated the eigenvalue density and level spacing of some random matrices and their asymptotic behavior as the matrix size tends to infinity. This investigations were continued by Dyson and several other authors. In the 80's the original Wigner-Dyson's model underwent a revival in the physical literature because of its connections with quantum chaos from one side and with the so–called *large N limit in Euclidean quantum field theory* in which one introduces a fictitious spin parameter in a real physical model and then studies the approximations of this model when the number of spin dimensions tends to infinity. This approach revealed the emergence of semicircular ensembles as limit eigenvalue distributions in a wide class of random matrices including *band random matrices*.

In the late 80's the study of large random matrices underwent a revival also in the mathematical literature because of its connections with the notion of free relation introduced by Voiculescu. It was realized that free relation describes the statistics of $N \times N$ random matrices in the limit $N \to \infty$ in the sense that large random matrices become asymptotically free in this limit.

The following typical result in this direction explains in what sense classical indepedence for $N \times N$ matrix valued random variables implies asymptotic freeness in the limit $N \to \infty$:

Let $T_n^{(1)}, \ldots, T_n^{(m)}$ be $n \times n$ random matrices with entries independent and gaussian with mean 0 and covariance $1/n$. Then, as $n \to \infty$, the m matrices $(T_n^{(1)}, T_n^{*(2)}, \ldots, T_n^{(m)*})$ converge, in the sense of moments of any order, to m free random varialbles $(T^{(1)}, T^{*(2)}), \ldots, T^{(m)*})$.

Now, the Gaussian central limit theorem for free random varialbles states that sums of free independent (classical) random varialbles with the standard central limit normalization converge to a classical random variable with semi–circle distribution.

Combining this free central limit theorem with the asymptotic freedom, in the limit $N \to \infty$, of classically indepedent $N \times N$ matrix valued random variables, one deduces that, for gaussian random matrices the distribution of eigenvalues goes to the Wigner semicircle law. This is usually expressed by saying that semicircular systems arise from asymptotic behaviour of large gaussian random matrices.

Similarly the Poisson central limit theorem for free random varialbles leads to the free Poisson distribution which coincides with the Marchenko–Pastur distribution (also called Uhlman distribution in statistics) whose density is:

$$MP_r(x) := \frac{\sqrt{4r - (x^2 - (1+r))^2}}{2\pi r \cdot x} \, dx$$

and which arises from limit theorems for triangular random matrices.

So in both cases the quantum probabilistic approach unveils the deeper root of the universl emergence of some special distributions in the asymptotics of large matrices, allows to recover several results on random matrices, obtained in classical probability, and in fact in several cases to go beyond these results.

In particular the logarithmic energy, introduced by Voiculescu as a free analogue of entropy was shown to control the large deviations of the large N limit of several matrix ensambles.

This justifies the extensive use of free probability theory in connection with the asymptotics of large random matrices.

However even now many interesting problems remain open, for example little is known on more detailed statistics of the eigenvalues in the large N limit, e.g., the spacing distributions which is related to the appearance of quantum chaos.

The development of quantum information and computer also contributed to the revival of interest in large random matrices as natural models of large arrays of logical gates and of the corresponding network flows.

The recent monograph of Hiai and Petz combines a synthesis of the most advanced developments with a clear and exhaustive exposition of the basic ideas and techniques of the theory.

The main lectures of the school will be devoted to a systematic exposition of the main topics dealt with in the above monograph. Several random matrix ensambles will be introduced, their joint eigenvalue distribution and limiting eigenvalue densities will be discussed. The relations between random matrices and free probability theory will be explained and random matrix models of noncommutative distributions, with the corresponding large deviation theorems, will be presented.

The main focus of the school will be on constructive aspects, examples and applications, based on essentially elementary mathematical tools.

Only in the last part of the school, and in the seminars which will complement the main lectures, more advanced tools and notions of oerator algebra will be used, but in an essentially self–contained manner.

Bibliography

F. Hiai and D. Petz: *The semicircle law, free random variables and entropy*, American Mathematical Society, Providence, (2000)

Main lecturers

Fumio Hiai (Tohoku University, Sendai, Japan)
Boris Khoruzenko (University of London, London, UK)
Dénes Petz (Budapest University of Technology, Budapest, Hungary)
Steen Thorbjørgsen (Odense University, Odense Danmark)