Entropy and Central Limit on the Algebra of the Canonical Commutation Relation *

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1. Introduction. A basic result of probability theory says that the standardized sum of independent identically distributed random variables with finite variance converges to a Gaussian distribution. When the convergence is inderstood in law this is the simplest form of the central limit theorem [9]. Under some additional assumption, however the density functions converge in L^1-norm as well [18]. In the last twenty years several analogues of the central limit theorem have been obtained in connection with the algebra of the canonical commutation relation. In this context the quasifree states showed up in place of the Gaussian distribution.

In the present paper first of all we reformulate the central limit theorem of Cushen and Hudson [7] in order to create a setting for the study of norm convergence. For a state φ of the algebra of the canonical commutation relation we define a weighted convolution

$$(\varphi \hat{+} \varphi \hat{+} \varphi \hat{+} \dots \hat{+} \varphi)\frac{1}{\sqrt{n}} \qquad \text{(n summands)} \tag{1}$$

which is a state of the algebra again. Due to the above mentioned central limit theorem the sequence (1) converges to a quasifree state pointwise. Compared with probability theory φ corresponds to the common distribution of the the independent random variables, while (1) corresponds to the distribution of the standardized sum. (This explains also the use of the terminology of weighted convolution.) We apply a relative entropy method and prove the norm convergence of the sequence (1) under the condition of finite entropy. In probability theory the same method was used by Barron [4]. However, we could not follow the lines of his agument in all respect, because the Fisher information and the related inequalities are not worked out in this setting. On the other hand, we do not restrict ourselves to finite degree of freedom.

During the central limit the entropy increases and among the states with the same two-point function the quasifree one has the maximal entropy. This is referred to as the maximum entropy principle in mathematical physics [23] and was also the subject of the paper [22] (in one degree of freedom). In this paper we show that the relative entropy with respect to the limiting quasi-free state goes to 0 provided it has a finite initial value.

2. The CCR-algebra and convolution of states. Let H be a real linear space. A bilinear from σ of H is called symplectic form if $\sigma(f,g) = -\sigma(g,f)$ for $f,g \in H$.

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By a symplectic space we mean a pair (H, σ) consisting of a linear space H and a symplectic form σ. To each symplectic space (H, σ) a C*-algebra $\mathrm{CCR}(H, \sigma)$ is associated which is strongly related to the Weyl form of the canonical commutation relation:

$$W(f)W(g) = W(f+g)\exp\left(-\tfrac{\mathrm{i}}{2}\sigma(f,g)\right) \tag{2}$$

The C*-algebra $\mathrm{CCR}(H, \sigma)$ is determined up to isomorphism by the following two conditions.

(i) $\mathrm{CCR}(H, \sigma)$ is generated by unitary operators $W(f)$ $(f \in H)$ satisfying the canonical commutation relation (2).
(ii) If a C*-algebra $\mathcal{A}$ contains unitaries $W'(f)$ $(f \in H)$ satisfying the relation (2) then there exists a homomorphism $\alpha : \mathrm{CCR}(H, \sigma) \to \mathcal{A}$ so that $\alpha(W(f)) = W'(f)$ $(f \in H)$.

Let us mention that in the extreme case when $\sigma \equiv 0$, the algebra $\mathrm{CCR}(H, \sigma)$ is isomorphic to the C*-algebra of all continuous functions on the compact space $\hat{H}$ where $\hat{H}$ is the compact dual group of the discrete abelian group H. General references on the CCR-algebra are [6, 8, 13, 15, 16].

Since the linear combinations of the Weyl unitaries $\{W(f) : f \in H\}$ is dense in $\mathrm{CCR}(H, \sigma)$, any state φ is determined by the function $f \to G(f) \equiv \varphi(W(f))$ $(f \in H)$ which is called the characteristic function of the state φ. It can be proven that $f \mapsto G(f)$ is the characteristic function of a state of $\mathrm{CCR}(H, \sigma)$ if and only if $G(0) = 1$ and the kernel

$$(f, g) \to G(f-g)\exp\left(\tfrac{\mathrm{i}}{2}\sigma(f,g)\right) \tag{3}$$

is positive, that is, for any $f_1, f_2, \dots, f_n \in H$ and $c_1, c_2, \dots, c_n \in \mathbb{C}$ we have

$$\sum_{j,k=1}^{n} c_j, \bar{c}_k G(f_j - f_k)\exp\left(\tfrac{\mathrm{i}}{2}\sigma(f_j, f_k)\right) \geq 0\,.$$

Now we define two operations for states of CCR algebras. For the states $\varphi_1 \in \Sigma(\mathrm{CCR}(H, \sigma_1))$ and $\varphi_2 \in \Sigma(\mathrm{CCR}(H, \sigma_2))$ a state $\varphi_1 \hat{+} \varphi_2$ on $\mathrm{CCR}(H, \sigma_1 + \sigma_2)$ is determined by the formula

$$(\varphi_1 \hat{+} \varphi_2)(W_{12}(f)) = \varphi_1(W_1(f))\varphi_2(W_2(f))\,. \tag{4}$$

The characteristic function of $\varphi_1 \hat{+} \varphi_2$ is the product of the characteristic functions of φ_1 and φ_2. Since the pointwise product of positive kernels is positive again [5], (4) really defines a state.

The other operation associates a state $\varphi \cdot \lambda$ of the algebra $\mathrm{CCR}(H, |\lambda|^2\sigma)$ to $\varphi \in \Sigma(\mathrm{CCR}(H, \sigma))$ and $\lambda \in \mathbb{R}$. We set

$$\varphi \cdot \lambda \left(W_{|\lambda|^2\sigma}(f)\right) = \varphi\left(W_\sigma(\lambda f)\right)\,. \tag{5}$$

It is checked readily that

$$(\varphi \hat{+} \psi) \cdot \lambda = \varphi \cdot \lambda \hat{+} \psi \cdot \lambda\,. \tag{6}$$

If $\varphi_i \in \Sigma(\mathrm{CCR}(H, \sigma))$ and $\sum_{i=1}^{n} |\lambda_i|^2 = 1$ then

$$\varphi_1 \cdot \lambda_1 \hat{+} \varphi_2 \cdot \lambda_2 \hat{+} \dots \hat{+} \varphi_n \cdot \lambda_n \in \Sigma(\mathrm{CCR}(H,\sigma)) \tag{7}$$

and

$$(\varphi_1 \cdot \lambda_1 \hat{+} \varphi_2 \cdot \lambda_2 \hat{+} \dots \hat{+} \varphi_n \cdot \lambda_n)(W(h)) = \prod_{i=1}^{n} \varphi_i(W(\lambda_i h)) \,. \tag{8}$$

The state (7) might be called the convolution of $\varphi_1, \varphi_2, \dots, \varphi_n$ with the weights $\lambda_1, \lambda_2, \dots,$
λ_n. We shall prove the convergence of the sequence

$$(\varphi \hat{+} \varphi \hat{+} \varphi \hat{+} \dots \hat{+} \varphi) \frac{1}{\sqrt{n}} \qquad \text{(n summands)}$$

which is a kind of central limit. (The normalization $\sum_i \lambda_i^2 = 1$ in the definition was prefered to convex combination in order to emphesise the strong relationship with central limit.)

The symplectic form σ is called nondegenerate if $\sigma(f,g) = 0$ for every $f \in H$ implies $g = 0$. If (H,σ) is a nondegenerate symplectic space then $\mathrm{CCR}(H,\sigma)$ is determined uniquely by condition (i) in its definiton. This result is the Slawny theorem [21].

Let $\mathcal{H}$ be a complex Hilbert space. Then $\sigma(f,g) = \mathrm{Im}\,\langle f,g\rangle$ is a nondegenerate symplectic form and we write $\mathrm{CCR}(\mathcal{H})$ for the corresponding C*-algebra of the canonical commutation relation. Now we describe the Fock representation of $\mathrm{CCR}(\mathcal{H})$.

We consider the full Fock space

$$\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \oplus \dots$$

and define for each $f \in \mathcal{H}$ two bounded operators $c(f)$ and $c^*(f)$.

$$\begin{aligned} &c(f)\Omega = 0 \\ &c(f)(g_1 \otimes \dots \otimes g_n) = \langle f, g_1\rangle g_2 \otimes \dots \otimes g_n \\ &c^*(f)\Omega = f \\ &c^*(f)(g_1 \otimes \dots \otimes g_n) = f \otimes g_1 \otimes \dots \otimes g_n \end{aligned} \tag{9}$$

The correspondence $f \mapsto c^*(f)$ is (complex) linear and c*(f) is the adjoint of $c(f)$. The relation

$$c(g)c^*(f) = \langle g, f\rangle I \qquad (g, f \in \mathcal{H}) \tag{10}$$

holds. The number operator N is a positive selfadjoint operator so that $\mathcal{H}^{(1)}, \otimes \dots \otimes \mathcal{H}^{(n)}$ is an eigensubspace corresponding to the eigenvalue n. Let P_+ be the projection of $\mathcal{F}(\mathcal{H})$ onto the symmetric (Bose) Fock space $\mathcal{F}_+(\mathcal{H})$ and set

$$\begin{aligned} a(f) &= P_+ c(f) N^{1/2} P_+ \,, \\ a^+(f) &= P_+ N^{1/2} c^*(f) P_+ \,. \end{aligned} \tag{11}$$

For the (Bose) annihilation and creation operators. They satisfy the canonical commutation relations:

$$\begin{aligned}[a^+(f), a^+(g)] = [a(f), a(g)] = 0 \\ [a(f), a^+(g)] = \langle f, g\rangle I\end{aligned} \tag{12}$$

for every $f, g \in \mathcal{H}$ (on the dense subspace of finite particle vectors). By means of the field operators

$$B(f) = \frac{1}{\sqrt{2}}\left(a(f) + a^+(f)\right) \qquad (f \in \mathcal{H}) \tag{13}$$

one defines the Weyl unitaries

$$W(f) = \exp(\mathrm{i}\, B(f)) \quad (f \in \mathcal{H})\,. \tag{14}$$

Since

$$W(f)W(g) = W(f+g)\exp\left(-\tfrac{\mathrm{i}}{2}\mathrm{Im}\,\langle f, g\rangle\right) \quad (f, g \in \mathcal{H}) \tag{15}$$

the Weyl unitaries acting on the Bose Fock space $\mathcal{F}_+(\mathcal{H})$ give an irreducible representation of the abstract C*-algebra $\mathrm{CCR}(\mathcal{H}, \sigma)$ when the symplectic form $\sigma(f, g) \equiv \mathrm{Im}\,\langle f, g\rangle$. Although $\mathrm{CCR}(\mathcal{H}, \sigma)$ admits different irreducible representations (for an infinite dimensional Hilbert space $\mathcal{H}$), the Fock representation is the most important one.

Let φ be a state of $\mathrm{CCR}(H, \sigma)$ and consider the corresponding GNS-representation π_φ on the Hilbert space $\mathcal{H}_\varphi$. If $t \mapsto \varphi(W(tf))$ is continuous for every $f \in H$ then the state φ is called regular and $t \mapsto \pi_\varphi(W(tf))$ is a strongly continuous unitary group. Due to Stone theorem it possesses a characteristic $B_\varphi(f)$ which is a selfadjoint operator on $\mathcal{H}_\varphi$. In most cases we restrict ourselves to states of $\mathrm{CCR}(\mathcal{H}, \sigma)$ such that $t \mapsto \varphi(W(tf))$ is at least twice differentiable. One can prove easily that in this case the cyclic vector $\Phi \in \mathcal{H}_\varphi$ is in the domain of the unbounded operator $B_\varphi(f)$ for every $f \in H$. If, in addition the state φ is even, that is, $\varphi(W(f)) = \varphi(W(-f))$ for every $f \in H$, then differentiating the relation

$$\langle e^{\mathrm{i}\, tB_\varphi(f)}\Phi, \Phi\rangle = \langle e^{-\mathrm{i}\, tB_\varphi(t)}\Phi, \Phi\rangle$$

we obtain

$$\langle B_\varphi(f)\Phi, \Phi\rangle = 0 \qquad (f \in H) \tag{16}$$

3. Relative entropy and entropy. The relative entropy of two states of an arbitrary C*-algebra was defined by Araki [2] by means of the relative modular operator in the GNS-representation. For our purposes Kosaki's equivalent variational formula will do [14].

$$\begin{aligned}&S(\omega, \varphi) \\ &= \sup_n \sup_x \left\{\omega(I)\log n - \int_{1/n}^{\infty} \omega(y(t)^*y(t)) + t^{-1}\varphi(x(t)x(t)^*)\frac{dt}{t}\right\}\end{aligned} \tag{17}$$

where the first sup is taken over all natural numbers n, the second one is over all step functions $x : (1/n, \infty) \to \mathcal{A}$ with finite range and $y(t) = I - x(t)$.

The following lemma is a consequence of Kosaki's formula.

Lemma 1. Let φ and ω be arbitrary states of $\mathrm{CCR}(\mathcal{H})$. Then

$$S(\psi,\varphi)=\sup\{S(\psi|\mathcal{A}\,,\ \varphi|\mathcal{A}):\mathcal{A}=\mathrm{CCR}(\mathcal{K}),\\ \mathcal{K}\subset\mathcal{H}\text{ is finite dimensional subspace}\}\,.$$

Proof. Let $\mathcal{B}\subset\mathrm{CCR}(\mathcal{H})$ be the *-algebra of finite linear combinations of Weyl operators. Then $\mathcal{B}$ is a norm dense subalgebra of $\mathrm{CCR}(\mathcal{H})$. The relative entropy $S(\psi,\varphi)$ can arbitrarily well approximated by an expression

$$\log m-\int_{1/m}^{\infty}t^{-1}\psi(y(t)^*y(t))+t^2\varphi(x(t)x(t)^*)\,dt \tag{18}$$

due to Kosaki's formula. One can choose $x(t),y(t)\in\mathcal{B}$ and in fact, (18) is a finite sum. If we take the subspace $\mathcal{K}$ which is generated by the symbols of the Weyl unitaries appearing in (18) then $S(\psi|\mathrm{CCR}(\mathcal{K}),\ \varphi|\mathrm{CCR}(\mathcal{K}))$ is a good approximation of $S(\psi,\varphi)$. □

The relative entropy is additive under (projective) tensor product and concerning its main properties we refer to the surveys [3, 17]. Below we we need the following fact. If $S(\omega,\varphi)$ is finite and π is a type I factor representation on a Hilbert space $\mathcal{K}$ such that φ admits a density in $B(\mathcal{K})$, that is,

$$\varphi(a)=\mathrm{Tr}\,D_\varphi\pi(a)\,,$$

then ω has a density D_ω as well and

$$S(\omega,\varphi)=\mathrm{Tr}\,D_\omega(\log D_\omega-\log D_\varphi)\,.$$

In particular, if ω has finite von Neumann entropy $S(\omega)=-\mathrm{Tr}\,D_\omega(\log D_\omega)$ then

$$S(\omega,\varphi)=-S(D_\omega)-\mathrm{Tr}\,(D_\omega\log D_\varphi)\,.$$

The relative entropy and the von Neumann entropy are weak* lower semicontinuous quantities. However, under the "boundedness of the energy" the entropy is continuous. More precisely, let $0\le H\in B(\mathcal{H})$ be a selfadjoint operator such that $\mathrm{Tr}\,e^{-\beta H}<+\infty$ for every $\beta>0$. Then for every $t>0$ the entropy is continuous on the set

$$\{\varphi\in\Sigma_*(B(\mathcal{H})):\quad \varphi(H)\le t\}\,.$$

This was proved in [25].

Finally, we recall the basic inequality

$$||\varphi-\psi||^2\le 2S(\varphi,\psi) \tag{19}$$

which will be used to prove norm convergence of states by estimating relative entropy (see [12]).

4. Strengthened central limit. The following proposition is a central limit theorem in the context of the algebra of the canonical commutation relation. It was proved by Cushen and Hudson [7], see also [20]. In the mean time several generalizaton was obtained, for exanple [1, 10, 11, 24]. The relation to the central limit theorem of probability theory is clear from the proof which is not repeated here here.

Proposition 2. Let φ be a twice differentiable even state of the algebra $\mathrm{CCR}(H,\sigma)$ over the symplectic space (H,σ). Then the weighted convolution

$$(\varphi\hat{+}\varphi\hat{+}\dots\hat{+}\varphi)\frac{1}{\sqrt{n}}$$

converges pointwise to a state φ_Q given by

$$\varphi_Q(W(f)) = \exp\left(-\frac{1}{2}||B_\varphi(f)\varPhi||^2\right) \qquad (f\in H)\,.$$

The state φ_Q appeared in Proposition 2 is called the quasi-free reduction of the state φ. A state φ is called quasi-free if $\varphi=\varphi_Q$. The previous central limit theorem shows that quasi-free states are analogous of Gaussian distributions. Using entropy methods we shall see that under some additional hypothesis the limit in the central limit theorem holds in norm.

From now on we consider the algebra of the canonical commutation relation over a complex pre-Hilbert space. Then the symplectic form $\sigma(f,g)=\mathrm{Im}\,\langle f,g\rangle$ is nondegenerate and $\mathrm{CCR}(\mathcal{H})$ is simple. By means of the complex structure of $\mathcal{H}$ one can define gauge invariant states and creation-annihilation operators. The state φ of $\mathrm{CCR}(\mathcal{H})$ is said to be gauge invariant if

$$\varphi(W(\lambda f)) = \varphi(W(f)) \quad (f\in\mathcal{H},\ \lambda\in\mathbb{C},\ |\lambda|=1)\,. \tag{20}$$

The creation and annihilation operators appear in the GNS-Hilbert space of a (regular) state φ:

$$\begin{aligned} a_\varphi(f) &= \frac{1}{\sqrt{2}}(B_\varphi)(f) + \mathrm{i}\,B_\varphi(\mathrm{i}t)\\ a_\varphi^+(f) &= \frac{1}{\sqrt{2}}(B_\varphi(f) - \mathrm{i}\,B_\varphi(\mathrm{i}t))\,. \end{aligned} \tag{21}$$

The state

$$\varphi_F(W(f)) = \exp\left(-\frac{1}{4}||f||^2\right) \quad (f\in\mathcal{H})$$

is a gauge invariant infinitely many times differentiable state of $\mathrm{CCR}(\mathcal{H})$ and called Fock state because the corresponding GNS-Hilbert space is naturally isomorphic to the Bose-Fock space over the Hilbert space $\mathcal{H}$. The Fock state is a pure quasi-free state. We shall mainly deal with twice differentiable gauge invariant states. Let now φ be such a state and $\varPhi$ be the cyclic vector of its GNS-representation. Then $\varPhi\in\mathcal{D}(a_\varphi(f))$, $\mathcal{D}(a_\varphi^+(f))$ for every $f\in\mathcal{H}$ and $\langle a_\varphi(g)\varPhi, a_\varphi(f)\varPhi\rangle$ is defined. Formally it can be written as $\varphi(a_\varphi^*(g)a_\varphi(f))$ which is called the two-point-function of the state φ. The two-point-function

$$t(f,g) = \varphi(a_\varphi^*(g)a_\varphi(f)) \tag{22}$$

is a positive sesquilinear form over the Hilbert space $\mathcal{H}$. To each positive sesquilinear form t there exists a state ω determined by the formula

$$\omega(W(f)) = \exp\left(-\frac{1}{4}||f||^2 - \frac{1}{2}t(f,f)\right) \quad (f \in \mathcal{H}) \tag{23}$$

which is a gauge invariant quasi-free state. From the relations

$$\omega(B_\omega(f)^2) = -\left.\frac{d^2\omega(W(tf))}{dt^2}\right|_{t=0} = \frac{1}{2}||f||^2 + t(f,f) \tag{24}$$

and

$$B_\omega(f)^2 = \frac{1}{2}\left(a^*_\omega(f)^2 + a_\omega(f) + 2a^*_\omega(f)a_\omega(f) + ||f||^2 I\right)$$

follows that the two-point-function of ω is $(f,g) \mapsto t(f,g)$. It is clear from Proposition 2 and the formula (24) that ω is the quasi-free reduction of φ. So one can describe the quasi-free reduction as follows. A (twice differentiable gauge invariant) state φ of CCR($\mathcal{H}$) determines a two-point-function (by (22)) and the quasi-free reduction of φ is the quasi-free state given by this two-point-function, see (23).

Above the weighted convolution $\varphi_1 \cdot \lambda_1 \hat{+} \varphi_2 \cdot \lambda_2$ was defined for real coefficients λ_1 and λ_2. If the test fuction space $\mathcal{H}$ has a complex linear structure then $\lambda_1, \lambda_2 \in \mathbb{C}$ may be allowed. However, for a gauge invariant state φ_1, we have $\varphi_1 \cdot \lambda_1 = \varphi_2 \cdot |\lambda_1|$.

The next proposition contains the basic entropy inequality for weighted convolution. The completeness of the Hilbert space $\mathcal{H}$ will not play any role, $\mathcal{H}$ could be a complex inner product space as well.

Proposition 3. Let $\varphi_1, \varphi_2 \in \Sigma(\mathrm{CCR}(\mathcal{H}))$ be gauge invariant states and set

$$\varphi = \frac{\varphi_1 \hat{+} \varphi_1 \hat{+} \ldots \hat{+} \varphi_1}{\sqrt{n+k}} \hat{+} \frac{\varphi_2 \hat{+} \ldots \hat{+} \varphi_2}{\sqrt{n+k}}$$

where $n, k \in N$ and φ_1 appears n times and φ_2 k times. Then

$$S(\varphi) \geq \frac{n}{n+k} S(\varphi_1) + \frac{k}{n+k} S(\varphi_2)$$

Proof. We consider $\mathcal{A}_1 = \mathcal{A}_2 = \mathrm{CCR}(\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H})$ with a direct sum of $(a+k)$-fold. $\mathcal{A}_i$ can be identified with the $(n+k)$-fold tensor product

$$\mathrm{CCR}(\mathcal{H}) \otimes \mathrm{CCR}(\mathcal{H}) \otimes \ldots \otimes \mathrm{CCR}(\mathcal{H}))$$

under the identification

$$W_i(h_1 \oplus \ldots \oplus h_{n+k}) \mapsto W(h_1) \otimes W(h_2) \otimes \ldots \otimes W(h_{n+k})\,.$$

(The CCR-algebras are known to be nuclear [8], hence the C*-tensor product is unique.) On the algebra $\mathcal{A}_2$ we consider the product state

$$\psi = \varphi_1 \otimes \ldots \otimes \varphi_1 \otimes \varphi_2 \otimes \ldots \otimes \varphi_2$$

where φ_2 appears in h factors and φ_2 in k ones. According to the additivity of the entropy we have

$$S(\psi) = nS(\varphi_1) + kS(\varphi_2)\,. \tag{25}$$

If an $(n+k) \times (n+k)$ unitary matrix is given then a Bogoliubov isomorphism α arises between $\mathcal{A}_1$ and $\mathcal{A}_2$ as follows.

$$\alpha : W_1(h_1 \oplus \ldots \oplus h_{n+k}) \mapsto W_2\left(\sum_{i=1}^{n+k} a_{1,i}h_i \oplus, \ldots, \oplus \sum_{i=1}^{n+k} a_{n+k,i}h_i\right)\,.$$

Let us compute the marginal of $\psi \circ \alpha$ on the j-th marginal $(1 \le j \le n+k)$.

$$\psi \circ \alpha(W_1(0 \oplus \ldots \oplus h^{(j)} \oplus \ldots \oplus 0)) = \psi(W_2(a_{1,j}h \oplus a_{2j} \oplus \ldots \oplus a_{n+k,j}h))$$
$$= \prod_{i=n+1}^{n} \varphi_1(W(a_{ij}h)) \prod_{i=n+1}^{n+k} \varphi_2(W(a_{ij}h))\,.$$

Choosing

$$(a_{ij}) = \frac{1}{\sqrt{n+k}} e^{2ij\pi/(n+k)} \quad (1 \le i,j \le n+k) \tag{26}$$

we have a unitary matrix (a_{ij}) and the j-th marginal is nothing else but φ. Refering to the subadditivity of the entropy we conclude

$$S(\psi) = S(\psi \circ \alpha) \le (n+k)S(\varphi)$$

which gives the proof with (25). □

It φ be the same as in the previous proposition and define ψ analogously, that is

$$\psi = \frac{\psi_1 \hat{+} \psi_1 \hat{+} \ldots \hat{+} \psi_1}{\sqrt{n+k}} \hat{+} \frac{\psi_2 \hat{+} \psi_2 \hat{+} \ldots \hat{+} \psi_2}{\sqrt{n+k}}$$

(where the states ψ_1 and ψ_2 occur n- and k-times, respectively.) Following the proof of Proposition 3 one can obtain a relative entropy version. Namely,

$$S(\varphi,\psi) \le \frac{n}{n+k} S(\varphi_1,\psi_1) + \frac{k}{n+k} S(\varphi_2,\psi_2)\,. \tag{27}$$

The direction of the inequality is the opposite because the relative entropy is superadditive under tensor product while the entropy is subadditive.

Another remark concerns the gauge invariance. If $n = k = 1$ then the matrix (26) is real and the proof works for even states.

Theorem 5. Let φ be a gauge invariant twice differentiable state of CCR($\mathcal{H}$) with quasi-free reduction ψ. If $S(\varphi,\psi) < +\infty$ then

$$S(\varphi_n,\psi) \to 0 \quad \text{and} \quad ||\varphi_n - \psi|| \to 0\,,$$

where

$$\varphi_n = \frac{\varphi \hat{+} \varphi \hat{+} \ldots \hat{+} \varphi}{\sqrt{n}} \quad (\varphi \quad \text{is } n \text{ times})\,.$$

Proof. We show that $S(\varphi_n,\psi) \to 0$. Due to Lemma 1 it is sufficient to see that

$$S(\varphi_n|\mathrm{CCR}(\mathcal{K})\,,\psi|\mathrm{CCR}(\mathcal{K}))\to 0$$

for any finite dimensional subspace $\mathcal{K}\subset\mathcal{H}$. We fix $\mathcal{K}\subset\mathcal{H}$ and denote by $\bar\varphi_n$, $\bar\varphi$, $\bar\psi$ the restriction of φ_n, φ, ψ to $\mathcal{A}\equiv\mathrm{CCR}(\mathcal{K})$, respectively. Note that $\bar\psi$ is the quasi-free reduction of $\bar\varphi$ and $\bar\varphi_n$ is the weighted convolution sequence formed from $\bar\varphi$. Proposition 2 tells us that $\bar\varphi_n\to\bar\psi$ pointwise.

We view $\mathcal{A}$ in the Fock representation on the Hilbert space $\mathcal{F}_+(\mathcal{K})$. The quasi-free state $\bar\psi$ is normal with respect to the Fock representation, that is it has a normal extension to $B(\mathcal{F}_+(\mathcal{K}))$ which has a density D. According to (27) the sequence $n\mapsto nS(\bar\varphi_n,\bar\psi)$ is subadditive, in particular $S(\bar\varphi_{2n},\bar\psi)\le S(\bar\varphi_n,\bar\psi)\le S(\varphi,\psi)<+\infty$. (The subadditivity yields that $\lim_n S(\bar\varphi_n,\bar\psi)$ exists and we are showing that it must be 0.) Since $S(\bar\varphi_n,\bar\psi)$ is finite, $\bar\varphi_n$ also has a normal extension to $B(\mathcal{F}_+(\mathcal{K}))$ with a density D_n. We have

$$S(\bar\varphi_n,\bar\psi)=\mathrm{Tr}\,D_n(\log D_n-\log D_n)=-S(\bar\varphi_n)-\mathrm{Tr}\,D\log D\,.$$

We shall prove that $-\mathrm{Tr}\,D_n\log D=S(\bar\psi)$ independently of n and $S(\bar\varphi_n)\to S(\bar\psi)$. Having these relations proved, we arrive at $S(\bar\varphi_n,\bar\psi)\to 0$ and $S(\varphi_n,\psi)\to 0$.

The two-point-function of $\bar\psi$ is given by a positive operator $T\in\mathcal{B}(\mathcal{K})$ in the form

$$\bar\psi(a^*(g)a(f))=\langle f,Tg\rangle\quad(f,g\in\mathcal{K})$$

Assume that the positive eigenvalues of T are $\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_n>0$ with corresponding eigenvectors $f_1,f_2,\ldots,f_n$. Let

$$\lambda_i=\frac{e^{-S_i}}{1-e^{-S_i}}\quad(1\le i\le n)$$

and

$$H=\sum_{i=1}^n S_i\,a(f_i)^*a(f_i)\,.$$

It is known that $D=C\exp(-H)$ where $C=1/\mathrm{Tr}\,\exp(-H)$ is the normalization. From this it is clear that $\omega(\log D)$ depends only on the two-point-function of ω, in particular $-\mathrm{Tr}\,D_n\log D=-\mathrm{Tr}\,D\log D=S(\bar\psi)$. To show $S(\bar\varphi_n)\to S(\psi)$ we refer to the continuity of the entropy at bounded energy. We have $\mathrm{Tr}\,\exp(-\beta H)<+\infty$ and

$$\bar\varphi_n(H)=\bar\psi(H)<+\infty$$

and the above cited result is applicable.

Now $S(\varphi_n,\psi)\to 0$ has been proved and the inequality

$$||\varphi_n-\psi||^2\le 2S(\varphi_n,\psi)$$

completes the theorem. □

Since $S(\varphi,\varphi_Q)=-S(\varphi)+S(\varphi_Q)\ge 0$, among the state, with a given two-point function the quasi-free state has the largest entropy. This is called sometimes as maximum entropy principle. The quasi-free reduction is a (nonaffine projection onto the set of quasi-free states.

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