Problems

1 Linear algebra

1. Solve by Gauss-elimination:

$$\begin{aligned} 5x + 4z + 2t &= 3 \\ x - y + 2z + t &= 1 \\ 4x + y + 2z &= 1 \\ x + y + z + t &= 0 \end{aligned}$$

2.

$$\begin{aligned} x_1 + ix_2 - x_3 &= -i \\ x_1 + (1+i)x_2 + 2ix_3 &= -2 + 2i \\ x_1 + (1-i)x_2 - 2ix_3 &= -2 - 2i \end{aligned}$$

3.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 0 \\ -x_1 + x_2 - 6x_4 + x_5 &= 0 \\ 3x_2 + x_3 + 5x_4 - x_5 &= 0 \\ 2x_3 - 7x_4 + 7x_5 &= 0 \end{aligned}$$

4. Determine the value of a in such a way that there is a nontrivial solution:

$$\begin{aligned} x_1 + ix_2 - x_3 &= 0 \\ x_1 + (1+i)x_2 + 2ix_3 &= 0 \\ x_1 + (1-i)x_2 + ax_3 &= 0 \end{aligned}$$

5. Compute the determinants:

(a) $\begin{vmatrix} i & i & i \\ i & 2+2i & i \\ i & i & 1+3i \end{vmatrix}$

(b) $\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix}$

(c) $\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$

6. Apply Cramer's rule to solve:

$$\begin{aligned} x + 2y + 5z &= 9 \\ 8x - 7y + 3z &= 6 \\ 2x + 5y + z &= 5 \end{aligned}$$

7. Find a maximal linearly independent subsystem (a base), and express the remaining vectors as a linear combination in this base:

$$\mathbf{a_1} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}, \quad \mathbf{a_2} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}, \quad \mathbf{a_3} = \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}, \quad \mathbf{a_4} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}.$$

8. Are the following triples of vectors linearly independent?

(a)

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -i \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1+i \\ 0 \\ 0 \\ i \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 0 \\ 0 \\ 2-i \\ 1 \end{bmatrix}.$$

(b)

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -i \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1+i \\ 0 \\ 0 \\ i \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 3+2i \\ 1 \\ 0 \\ i \end{bmatrix}.$$

9. Is it possible to express $\mathbf{b}$ as a linear combination of $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$? If yes, determine the coefficients.

$$\mathbf{a_1} = \begin{bmatrix} i \\ -1 \\ -i \end{bmatrix}, \quad \mathbf{a_2} = \begin{bmatrix} 1+i \\ 2i \\ -2+2i \end{bmatrix}, \quad \mathbf{a_3} = \begin{bmatrix} 1-i \\ -2i \\ -2-2i \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ i \\ 1+i \end{bmatrix}.$$

10. Consider the base $B = \{\mathbf{v_1}, \mathbf{v_2}\}$ in $\mathbb{R}^2$, where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) Determine the coordinates of the vector $\mathbf{w} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$ in the base B.

(b) The vector $\mathbf{u}$ in base B has coordinates: $[\mathbf{u}]_B = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Determine the coordinates of $\mathbf{u}$ in the natural base.

11. Consider the two bases $B = \{\mathbf{u_1}, \mathbf{u_2}\}$, $B' = \{\mathbf{v_1}, \mathbf{v_2}\}$ in $\mathbb{R}^2$, where

$$\mathbf{u_1} = \begin{bmatrix} 15 \\ -2 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}, \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$[\mathbf{a}]_B = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$. Determine $[\mathbf{a}]_{B'}$.

12. Consider in $\mathbb{R}^3$ the linearis transformation A that projects any vector orthogonally onto the plane xy. Determine the matrix of A in the natural base.

13. Determine the matrix of the three dimensional linear transformation that maps the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of the natural base to

$$\mathbf{a_1} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}, \quad \mathbf{a_2} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{a_3} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix},$$

respectively.

14. Determine the matrix of the following linear transformation (in the natural base): $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3)$.

15. Let A be the linear transformation that maps the vectors $\mathbf{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to the vectors $\mathbf{y_1} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \mathbf{y_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{y_3} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, respectively. Determine the matrix of A in the base $\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$.

16. Let the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ in the base $B = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \end{bmatrix} \right\}$ have matrix $\begin{bmatrix} 3 & 1 \\ -2 & 8 \end{bmatrix}$. Determine the matrix of T in the

    (a) natural base,

    (b) in the base $B' = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 13 \end{bmatrix} \right\}$.

17. Determine the matrices of the orthogonal projections in $\mathbb{R}^3$

    (a) onto the plane S with equation $5x - 6y + z = 0$,

    (b) onto the plane T with equation $x + 5y - 2z = 0$.

    Compute the coordinates of the orthogonal projection of the point $P = (8, 1, 3)$ onto S and onto T.

18. Determine the eigenvalues and the eigenvectors of the following matrices:

    (a) $\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

    (b) $\begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

    (c) $\begin{bmatrix} -1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. Diagonalize the matrix $A = \begin{bmatrix} 7 & -2 \\ -2 & 3 \end{bmatrix}$.

20. Determine the spectral decomposition of $A = \begin{bmatrix} -3 & 5\sqrt{3} \\ 5\sqrt{3} & 7 \end{bmatrix}$.

21. Determine an orthonormal base for the subspace in $\mathbb{R}^4$ spanned by the vectors $\mathbf{a_1} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} 4 \\ 6 \\ 5 \\ 1 \end{bmatrix}, \mathbf{a_3} = \begin{bmatrix} 5 \\ 4 \\ -2 \\ 0 \end{bmatrix}$.

22. Determine the (at most) three degree polynomial that fits best to the points $(-3,2),(-1,4),(1,3),(2,0)$.

23. Apply the method of least squares to determine the line that fits best

    (a) the points $(-2,6),(-1,5),(0,1),(3,-4)$,

    (b) the points $(-2,0),(1,7),(2,10),(3,10),(4,17)$.

24. Determine the approximate solution of the linear system $Ax=b$ in the sense of least square error, and calculate the square error, if

    (a) $A=\begin{bmatrix} 3 & 4 \\ -2 & -5 \\ 1 & -2 \end{bmatrix}$, $b=\begin{bmatrix} 10 \\ -6 \\ 3 \end{bmatrix}$,

    (b) $A=\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -3 & 4 \\ -1 & 1 \end{bmatrix}$, $b=\begin{bmatrix} 4 \\ -3 \\ 2 \\ 1 \end{bmatrix}$.

25. Determine the singular value decomposition of the matrices

    (a) $A=\begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix}$

    (b) $B=\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

    (c) $C=\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

    .

26. Determine the polar decomposition of the matrices A and B from the preceding problem.

27. Let $A=\begin{bmatrix} -\frac{13}{5} & \frac{14}{5} \\ \frac{14}{5} & \frac{8}{5} \end{bmatrix}$, $B=\begin{bmatrix} -13 & 20 \\ -\frac{15}{2} & 12 \end{bmatrix}$. Determine the matrices e^{2A} and e^{-3B}.

2 Partial differential equations

28. Determine the Fourier series for the following 2π-periodic functions:

    (a) $f(x)=\begin{cases} 1 & \text{, ha } 0<x<\pi \\ 0 & \text{, ha } x\in\{-\pi,0,\pi\} \\ -1 & \text{, ha } \pi<x<0 \end{cases}$

    (b) $g(x)=|\sin x|,\ -\pi\le x\le\pi$

    (c) $h(x)=\pi-\frac{x^2}{\pi}$

29. Detemine the Fourier series for the 8-periodic $f(x)=\begin{cases} 2 & \text{, if } -4\le x<-2 \\ 0 & \text{, if } -2\le x<2 \\ 2 & \text{, if } 2\le x<4. \end{cases}$

30. Determine the Fourier sine series for $f(x)=\frac{\pi-x}{2}$ on $[0,\pi]$.

31. Determine the Fourier sine series for $f(x)=x^2$ on $[0,2]$.

32. For the problems below apply Bernoulli's method and refer to the Fourier series computed in class.

(a) $\begin{cases} u_{tt} = u_{xx}, & 0 < x < \pi,\ 0 < t \\ u(x,0) = \frac{\pi - x}{2}, & 0 \le x \le \pi \\ u_t(x,0) = 0, & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0, & 0 \le t \end{cases}$

(b) $\begin{cases} u_{tt} = 2u_{xx}, & 0 < x < 5,\ 0 < t \\ u(x,0) = 3x, & 0 \le x \le 5 \\ u_t(x,0) = \cos(\frac{3\pi}{5}x)\sin(\pi x), & 0 < x < 5 \\ u(0,t) = u(5,t) = 0, & 0 \le t \end{cases}$

(c) $\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < 2,\ 0 < t \\ u(x,0) = \sin(\frac{3\pi}{2}x), & 0 \le x \le 2 \\ u_t(x,0) = x^2, & 0 < x < 2 \\ u(0,t) = u(2,t) = 0, & 0 \le t \end{cases}$

For the following problems apply d'Alembert method.

33. A string extends along the interval $[0,6]$ with its to endpoints fixed, its vibration is described by the PDE $u_{tt} = 4u_{xx}$. The string is fixed at the points $(2,3)$ and $(3,3)$ and released at the time moment $t = 0$. Determine the shape of the string at $t = 1,5$.

34. A string extends along the interval $[0,2]$ with its to endpoints fixed, its vibration is described by the PDE $u_{tt} = 16u_{xx}$. Initially the string is at rest, while its velocity at $t = 0$ is according to $g(x) = 3x - 2$ $(0 \le x \le 2)$. Determine the shape of the string at $t = \frac{1}{4}$.

35. An infinite string is fixed at the points $(-5,0)$, $(-3,2)$, $(1,2)$, $(2,0)$ and released at $t = 0$. It evolves according to the PDE $u_{tt} = u_{xx}$. Determine its shape at $t = 5$.

36. The motion of an infinite string is given by:

$$\begin{cases} u_{tt} = 4u_{xx} \\ u(x,0) = \cos(3x) \\ u_t(x,0) = \frac{6x}{x^2+1} \end{cases}$$

Determine its shape at t.

37. Solve the following heat transport problems:

(a)

$$\begin{cases} u_t = 4u_{xx}, \quad 0 \le x \le 2 \\ u(x,0) = \sin(4\pi x)\cos\left(\frac{5\pi}{2}x\right) \\ u(0,t) = u(2,t) = 0 \end{cases}$$

(b)

$$\begin{cases} u_t = 3u_{xx}, \quad 0 \le x \le 3 \\ u(x,0) = x(3-x) \\ u(0,t) = u(3,t) = 0 \end{cases}$$

3 Vector calculus

38. Compute the line integrals of the vector fields.

(a) $\mathbf{F}(x,y,z) = (y^2 - x^2, 2yz, -x^2)$; $\mathbf{r}(t) = (t, t^2, t^3)$, $0 \le t \le 1$.

(b) $\mathbf{F}(x,y,z) = (y, z, x)$; $\mathbf{r}(t) = (2\cos t, 2\sin t, 3t)$, $0 \le t \le 2\pi$.

(c) $\mathbf{F}(x,y,z) = (z - y, x - z, y - x)$; integrate along: $A(0,0,0) \to B(1,1,1)$.

(d) $\mathbf{F}(x,y,z) = (x^2+y^2, x^2-y^2, 0)$; along the line $y = 2-3x$, $z = 0$, $0 \le x \le 1$.

(e) $\mathbf{F}(x,y,z) = (xy, yz, xz)$; integrate along: $A(1,0,0) \to B(0,0,1) \to C(0,1,0)$.

(f) $\mathbf{F}(x,y,z) = (xz+y, yz-x, -x^2-y^2)$; integrate along: $x^2+y^2 = 1$, $z = 3$ oriented counterclockwise.

39. Determine the following surface integrals. (For closed surfaces, use outward, otherwise upward orientation.)

(a) $\mathbf{F}(x,y,z) = (x,y,0)$; $\mathbf{r}(u,v) = (4\cos u\cos v, 4\cos u\sin v, 4\sin u)$, $0 \le u \le \frac{\pi}{2}$, $0 \le v \le \frac{\pi}{2}$.

(b) $\mathbf{F}(x,y,z) = (xy, 2x+y, z)$; $\mathbf{r}(u,v) = (u+2v, -v, u^2+3v)$, $0 \le u \le 3$, $0 \le v \le 1$.

(c) $\mathbf{F}(x,y,z) = (x,y,z)$; the surface is the torus obtained by rotating the unit circle inside the xz plane with center $(3,0)$ about the z axis; and constrain to its part lying above the plane $z = 0$.

40. Compute the divergence and the curl:

(a) $\mathbf{F}(x,y,z) = (xy, xz, yz)$

(b) $\mathbf{F}(x,y,z) = (\frac{x}{y}, \frac{y}{z}, xz)$

(c) $\mathbf{F}(x,y,z) = (\sin(x^2yz), \sin(xy^2z), \sin(xyz^2))$

(d) $\mathbf{F}(\mathbf{r}) = |\mathbf{r}|\mathbf{a}$ ($\mathbf{a}$ is some constant vector)

41. Apply curl-test to decide (if possible) whether the following vector fields are conservative. If the answer is yes, determine the potential.

(a) $\mathbf{F}(x,y,z) = (\frac{1}{x+y+z}, \frac{1}{x+y+z}, \frac{1}{x+y+z})$

(b) $\mathbf{F}(x,y,z) = (\cos(x+y)-\sin(x-y), \cos(x+y)+\sin(x-y), 0)$

(c) $\mathbf{F}(x,y,z) = (2xy-z^2-yz, x^2+z^2-xz, 2yz-2xz-xy)$

42. Compute the line integral of $\mathbf{F}(x,y,z) = (y+z, z+x, x+y)$ along $\mathbf{r}(t) = (a\sin^2 t, 2a\sin t\cos t, a\cos^2 t)$, $0 \le e \le \frac{\pi}{4}$. (Consider the potential.)

43. Apply Gauss theorem to compute the following surface integrals.

(a) $\mathbf{F}(x,y,z) = (x^3, y^3, z^3)$, the surface: $x^2+y^2+z^2 = a^2$.

(b) $\mathbf{F}(x,y,z) = (xy, yz, zx)$, on the surface of the brick $-2 \le x \le 7$, $3 \le y \le 4$, $-8 \le z \le -1$ with its face lying inside the plane $x = -2$ removed.

44. Apply Stokes theorem to compute the line integral of the vector field $\mathbf{F}(x,y,z) = (x^2y+e^x\sin z, \frac{x^3}{3} - e^{yz}, \sin(xyz))$ along the circle of radius 6 with center $(3,2)$ lying inside the xy plane.

45. Apply Green's theorem to compute the line integral of the vector field $\mathbf{F}(x,y) = (y^2, 3xy)$ along the semicircle of unit radius about the origin inside the upper half plane, oriented counterclockwise.

46. Apply Green's theorem to compute the area of the domain delimited by the x axis and the curve $\mathbf{r}(t) = (R(t-\sin t), R(1-\cos t))$; $0 \le t \le 2\pi$ (cycloid).