1. $c=1,\ L=\pi \quad f(x)=\sin 3x+\sin 5x-\sin 7x \quad g(x)=\begin{cases}4 & 0<x<\pi\\ 0 & x=0 \vee x=\pi\end{cases}$ 3p

$g(x)=\frac{16}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\dots\right)$ 2p

$u(x,t)=\sin 3x\cos 3t+\sin 5x\cos 5t-\sin 7x\cos 7t+$ 4p

$+\frac{16}{\pi}\left(\sin x\sin t+\frac{1}{9}\sin 3x\sin 3t+\frac{1}{25}\sin 5x\sin 5t+\dots\right)$ 6p

2. $\begin{pmatrix}1&2&1&1\\2&4&0&4\\-2&-4&1&-5\end{pmatrix}\sim\begin{pmatrix}1&2&1&1\\0&0&-2&2\\0&0&3&-3\end{pmatrix}\sim\begin{pmatrix}1&2&1&1\\0&0&1&-1\\0&0&0&0\end{pmatrix}$ 8p

$\operatorname{rank}A=2$ (vezéregyesek száma) 2p

$\operatorname{nullity}A=4-\operatorname{rank}A=4-2=2$ 2p

$\operatorname{rank}A^T=\operatorname{rank}A=2 \Rightarrow \operatorname{nullity}A^T=3-\operatorname{rank}A^T=3-2=1$ 3p

3. $0=\det(A-\lambda I)=(10-\lambda)^2-36=\lambda^2-20\lambda+64=(\lambda-4)(\lambda-16)$

$\lambda_2=4,\ \lambda_1=16 \Rightarrow$ pozitív def., mert sajátérték pozitívak 5p

$\underline{\lambda_2=4}$: $\begin{pmatrix}6&6\\6&6\end{pmatrix}\sim\begin{pmatrix}1&1\\0&0\end{pmatrix} \quad v_2=\begin{pmatrix}-1/\sqrt2\\1/\sqrt2\end{pmatrix}$ 2p

$\underline{\lambda_1=16}$ $\begin{pmatrix}-6&6\\6&-6\end{pmatrix}\sim\begin{pmatrix}1&-1\\0&0\end{pmatrix} \quad v_1=\begin{pmatrix}1/\sqrt2\\1/\sqrt2\end{pmatrix}$ 2p

$Q=\begin{pmatrix}1/\sqrt2&-1/\sqrt2\\1/\sqrt2&1/\sqrt2\end{pmatrix} \quad D=\begin{pmatrix}16&0\\0&4\end{pmatrix} \Rightarrow \sqrt{D}=\begin{pmatrix}4&0\\0&2\end{pmatrix}$ 2p

$B=Q\sqrt{D}Q^T=\begin{pmatrix}1/\sqrt2&-1/\sqrt2\\1/\sqrt2&1/\sqrt2\end{pmatrix}\begin{pmatrix}4&0\\0&2\end{pmatrix}\begin{pmatrix}1/\sqrt2&1/\sqrt2\\-1/\sqrt2&1/\sqrt2\end{pmatrix}=\frac12\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}4&4\\-2&2\end{pmatrix}=$

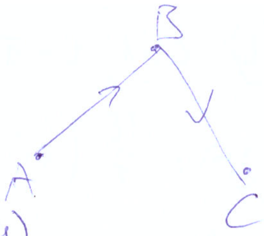
$=1/2\begin{pmatrix}6&2\\2&6\end{pmatrix}=\begin{pmatrix}3&1\\1&3\end{pmatrix}$ 4p

4. (a) $\operatorname{curl}F=\begin{vmatrix}i&j&k\\ \partial_x&\partial_y&\partial_z\\ 3x^2y+z^2\cos x & 2+x^3 & 2z\sin x+6\end{vmatrix}=i\,0-j(2z\cos x-2z\cos x)+$

$+k(3x^2-3x^2)=0$

és mindenhol sima $\Rightarrow$ potenciálos 7p

(b) $\frac{\partial U}{\partial x} = 3x^2y + z^2\cos x \Rightarrow U(x,y,z) = x^3y + z^2\sin x + U_1(y,z)$



$\frac{\partial U}{\partial y} = 2 + x^3 \Rightarrow U(x,y,z) = x^3y + 2y + U_2(x,z)$

$\frac{\partial U}{\partial z} = 2z\sin x + 6 \Rightarrow U(x,y,z) = z^2\sin x + 6z + U_3(x,y)$

így $U(x,y,z) = x^3y + z^2\sin x + 2y + 6z + C_0$ (8p)

$\int_\gamma \underline{F}\,d\underline{r} = U(C) - U(A) = 4 - 6 + C_0 - (2 + 6 + C_0) = 4 - 14 = \boxed{-10}$

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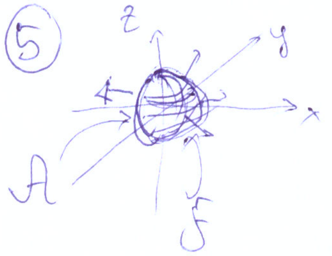
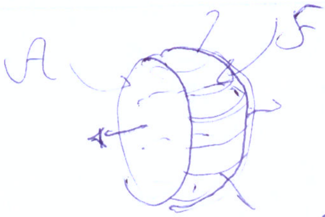
(5)





$S = A \cup F = \partial K$

ahol K a félgömb (tömör) (3p)

Gauss $\Rightarrow \iint_S \underline{G}(\underline{r})\,d\underline{A} = \iiint_K \operatorname{div}\underline{G}\,dx\,dy\,dz = 1 \cdot \operatorname{Vol} K = \frac{2\pi}{3}$ (4p)

$\operatorname{div}\underline{G} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} = 0 - 2 + 3 = 1$ (3p)

így: $\iint_F \underline{G}(\underline{r})\,d\underline{A} = \frac{2\pi}{3} - \iint_A \underline{G}(\underline{r})\,d\underline{A} = \frac{2\pi}{3} - \underbrace{\underline{n}\cdot\underline{G}}_{\text{const}}\cdot \operatorname{Area}(A) = \frac{2\pi}{3} - (-3)\cdot\pi =$

A normálisa (egységnyi): $(-1, 0, 0) = \underline{n}$

$= \boxed{\frac{11\pi}{3}}$ (5p)

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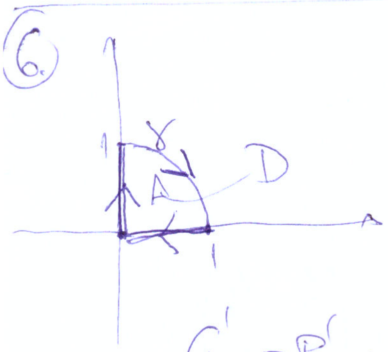
(6)



Green: $\int_\gamma \underline{G}(\underline{r})\,d\underline{r} = -\int_{-\gamma} \underline{G}(\underline{r})\,d\underline{r} = -\iint_D (Q'_x - P'_y)\,dx\,dy$

orientációval ellentétes (3p)

$Q'_x - P'_y = y^2 + 6x^2 - (-4x^2 - 9y^2) = 10x^2 + 10y^2$ ~~4p~~ (5p)

$\int_\gamma \underline{G}(\underline{r})\,d\underline{r} = -\iint_D 10(x^2+y^2)\,dx\,dy = (-10)\int_0^1\int_0^{\pi/2} r^2\, r\,d\varphi\,dr =$

$= (-5\pi)\int_0^1 r^3\,dr = -5\pi\left[\frac{r^4}{4}\right]_{r=0}^{r=1} = \boxed{-\frac{5\pi}{4}}$ (7p)