(1) $u_{tt} = u_{xx}$ $\quad u(x,t) = ?$ $\quad 0 < x < \pi, \quad 0 \le t$

$c = 1 \qquad L = \pi$

$u(x,0) = f(x) = \sin 3x - \sin 5x \Rightarrow f(x) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{L}\right)$

$A_3 = 1$; $A_5 = -1$ minden más k-ra $A_k = 0$

$u_t(x,0) = g(x) = \begin{cases} 7 & 0 < x < \pi \\ 0 & x = 0 \vee x = \pi \end{cases}$

$\Downarrow$

$$g(x) = \sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi x}{L}\right) = \frac{28}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right)$$

$D_1 = \frac{28}{\pi} \quad D_2 = 0 \quad D_3 = \frac{28}{3\pi} \quad D_4 = 0 \dots$

$D_k = \alpha_k B_k \quad \alpha_k = \frac{k\pi c}{L} = k$

$$u(x,t) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{k\pi c t}{L}\right) + \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi c t}{L}\right)$$

$$= \sin 3x \cos 3t - \sin 5x \cos 5t + \frac{28}{\pi}\left(\sin x \sin t + \frac{1}{9}\sin 3x \sin 3t + \frac{1}{5^2}\sin 5x \sin 5t + \dots\right)$$

(2)

$$\begin{pmatrix} -1 & 2 & 1 & 0 \\ -2 & 4 & 1 & 1 \\ 2 & -4 & -3 & 0 \\ 1 & -2 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} 4 & & 4 & 4 \end{matrix}$$

3 vezéregyes ⇒

$\dim(\mathrm{col}A) = 3$

↳ 1. 3. és 4. oszlop pivot

$$\begin{pmatrix} -1 \\ -2 \\ 2 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \\ -3 \\ 0 \end{pmatrix} \text{ és } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$ bázis $\mathrm{col}A$-ban.

(3)

$$\begin{pmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{pmatrix} \qquad 0 = \begin{vmatrix} 5/2 - \lambda & 3/2 \\ 3/2 & 5/2 - \lambda \end{vmatrix} = \left(5/2 - \lambda\right)^2 - 9/4 = \frac{25}{4} + \lambda^2 - 5\lambda - \frac{9}{4} =$$

$$= \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) \qquad \lambda_1 = 1 \qquad \lambda_2 = 4$$

A poz. def.

⇑

mivel pozitív sajátértékei vannak

sajátvektorok:

$\lambda_1 = 1 \qquad \begin{pmatrix} 5/2-1 & 3/2 \\ 3/2 & 5/2-1 \end{pmatrix} = \begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} t \\ -t \end{pmatrix} = t \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$\lambda_2 = 4 \qquad \underline{v}_2 = t \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$A = Q\,D\,Q^T \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$B = Q\,\sqrt{D}\,Q^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$

$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}$

(4)

$$\underline{F}(\underline{r}) = \left(2xy + 2xz,\ x^2 + z,\ x^2 + y\right)$$

$$\operatorname{curl} \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy+2xz & x^2+z & x^2+y \end{vmatrix} = \underline{i}(1-1) - \underline{j}(2x-2x) + \underline{k}(2x-2x) = 0$$

és $\underline{F}$ mindenhol sima $\Rightarrow$ $\underline{F}$ potenciálos (azaz konzervatív)

mi a potenciál?

$$\frac{\partial U}{\partial x} = 2xy + 2xz \Rightarrow U(x,y,z) = x^2 y + x^2 z + C_1(y,z)$$

$C(0,0,0)$
$D(0,1,1)$

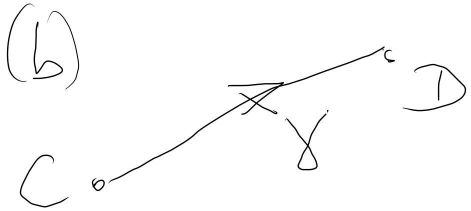
$$x^2 + \frac{\partial C_1}{\partial y} = \frac{\partial U}{\partial y} = x^2 + z \Rightarrow C_1(y,z) = yz + C_2(z)$$

$$\hookrightarrow U(x,y,z) = x^2 y + x^2 z + yz + C_2(z)$$

$$x^2 + y + \frac{\partial C_2}{\partial z} = \frac{\partial U}{\partial z} = x^2 + y \Rightarrow C_2(z) = C$$

$$U(x,y,z) = x^2 y + x^2 z + yz + C$$

(b)

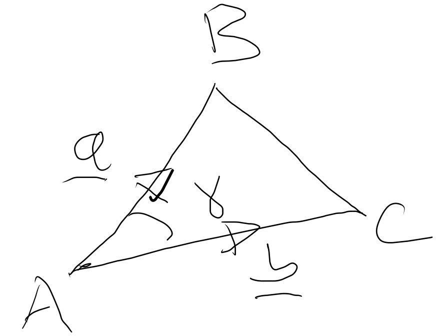


mivel potenciálos:

$$\int_\gamma \underline{F}\, d\underline{r} = U(D) - U(C) = 1 + C - C = \boxed{1}$$

5) $A(3,1,7)$ $B(5,1,7)$ $C(3,4,7)$

$$\boxed{\underline{F} = (-3,2,1)}$$



$\underline{a} = \overrightarrow{AB} = (2,0,0)$

$\underline{b} = \overrightarrow{AC} = (0,3,0)$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{vmatrix} = \underline{k} \cdot 6 = (0,0,6)$$

a háromszög területe $\frac{1}{2} |\underline{a}| \cdot |\underline{b}| \sin\gamma = \frac{|\underline{a} \times \underline{b}|}{2} = \frac{6}{2} = 3$

a háromszög egységnormálisa: $\boxed{\underline{n}} = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \boxed{(0,0,1)}$

$$\iint_F \underline{F}\, d\underline{A} = \text{felszín} \cdot \underline{n} \cdot \underline{F} = 3 \cdot 1 = \boxed{3}$$

felfelé mutat

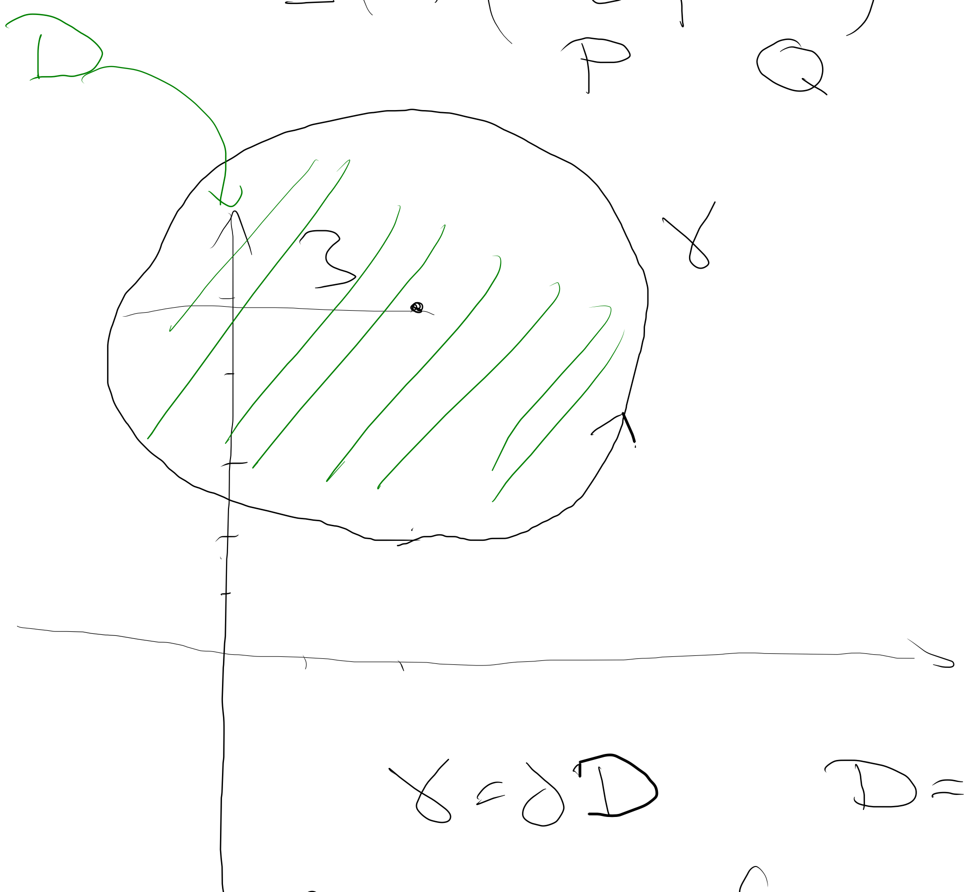
$$= \frac{(\underline{a} \times \underline{b}) \cdot \underline{F}}{2}$$

6.

$\underline{G}(\underline{r}) = \left(\underset{P}{7y} \, ; \, \underset{Q}{10x} \right)$

$(x-2)^2 + (y-5)^2 = 9 \; (= 3^2)$

γ zárt görbe mentén ((2,5) pont körüli 3 sugarú körvonal) óramutató járásával ellentétesen irányítva



$\gamma = \partial D \qquad D =$ körlap

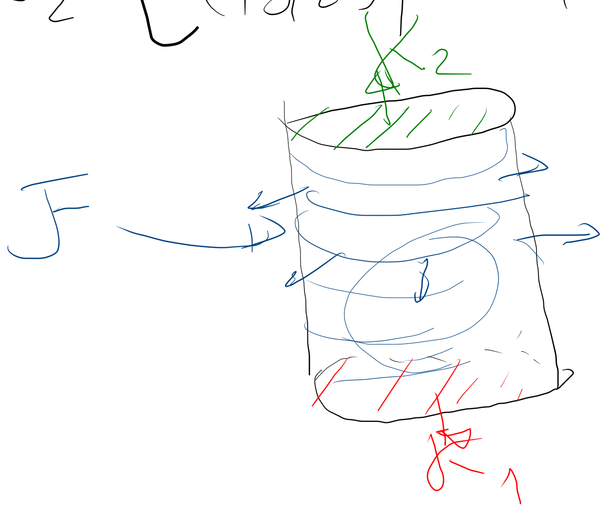
<u>Green</u>: $\displaystyle\oint_\gamma \underline{G}\, d\underline{r} = \iint_D \underbrace{(Q_x - P_y)}_{10-7} \, dx\,dy = 3 \cdot \text{Area}(D) =$

$= 3 \cdot 3^2 \pi = 27\pi$

$K_2 = \{(x,y,z) \mid x^2+y^2 \le 1/9,\ z=1\}$

F: henger palást

$\underline{G}\big|_{K_2} = (*, *, 0)$



$$\iint_F \underline{G}\, d\underline{A} = ?$$

$\underline{G}\big|_{K_1} = (*, *, 1)$

$$\underline{G} = \left(xz + e^y \cos z,\ yz + \sin(x^3+z^4),\ 1-z^2\right)$$

$S = F \cup K_1 \cup K_2 = \partial(H)$ $\quad H$ a tömör henger (mint test)

Gauss: $$0 = \iiint_H \operatorname{div} \underline{G}\, dx\,dy\,dz = \iint_S \underline{G}\, d\underline{A} = \iint_F \underline{G}\, d\underline{A} + \iint_{K_1} \underline{G}\, d\underline{A} + \iint_{K_2} \underline{G}\, d\underline{A}$$

$$\operatorname{div} \underline{G} = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z} = z + z - 2z = 0$$

így: $$\iint_F \underline{G}\, d\underline{A} = -\iint_{K_1} \underline{G}\, d\underline{A} - \underbrace{\cancel{\iint_{K_2} \underline{G}\, d\underline{A}}}_{=0} =$$

$\underline{n} =$

$$\iint_{K_1} \underline{G}\, d\underline{A} = \text{Area}(K_1)\cdot \underline{n}\cdot \underline{G} = -\frac{1}{9}\pi$$

K_1 $\qquad \underline{n} = (0, 0, -1)$

$\underline{G}\big|_{K_1} = (x, y, 1)$ $\qquad \underline{n}\cdot\underline{G} = \boxed{-1}$

konstans K_1-n

így $$\iint_{F} \underline{G}\, d\underline{A} = -\iint_{K_1} \underline{G}\, d\underline{A} = \boxed{\frac{\pi}{9}}$$

1. $u_{tt} = 4 u_{xx}$ végtelen húr $\boxed{c=2}$ $\qquad u(-1,2)=?$

$x \quad t$

$f(x) = u(x,0) \equiv 0 \quad g(x) = u_t(x,0) = \begin{cases} 3 & 0 \le x \le 10 \\ 0 & \text{egyébként} \end{cases}$

$$u(x,t) = \cancel{\frac{f(x+ct) + f(x-ct)}{2}} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

$$u(-1,2) = \frac{1}{2\cdot 2}\int_{-1-4}^{-1+4} g(s)\,ds = \frac{1}{4}\int_{-5}^{3} g(s)\,ds =$$

c

$$= \frac{1}{4}\int_{0}^{3} 3\,ds = \boxed{\frac{9}{4}}$$