High dimensional generalization of standard pairs and the coupling technique

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May 16, 2016

Abstract

In this paper we prove that a piecewise C^2, uniformly hyperbolic map of a d-dimensional manifold $\mathcal{M}$ ($d > 2$) enjoys exponential decay of correlations. This result is not new (first proved by Chernov; Discrete Contin. Dyn. Syst., **5** (1999), 425–448.), the novelty lies in the method we use. It is the high dimensional generalization of the coupling of standard pairs. Its advantage compared to previous techniques is that instead of using a symbolic coding of the dynamics it is a direct construction on the phase space, hence it is ideally suited to studying perturbations. This is the first time that this technique has been developed for systems with singularities in dimension greater than 2.

1 Introduction

An important aim in the theory of dynamical systems is to study strong statistical properties of hyperbolic maps, in particular the decay of correlations. Results from the recent decades provided exponential bounds on correlations for smooth hyperbolic maps in high dimensions (???) and also for hyperbolic maps with discontinuities (???). These properties are of particular interest not just for themselves, but also when considering more complex models of chaotic individual systems being coupled. Such models naturally occur when studying physically relevant examples, for instance models of heat conduction (see [12] via [11]) or models of molecules submerged into a gas [9]. To study these it is useful to develop methods that can be used to prove statistical properties of the individual systems and also flexible under perturbations. One such approach is the coupling of measures, a technique borrowed from probability theory. Its was first used by Young [16] to prove exponential decay of correlations for hyperbolic maps that can be modelled by a Young-tower. The later is a certain symbolic coding of the dynamics and hence, even though being very powerful, it is quite sensitive under perturbations. Later Bressaud and Liverani used this method in [3] to recover Bowen's results on Anosov maps of arbitrary dimensional manifolds. Dolgopyat further developed the method in (???) for the smooth case and introduced the so called standard pairs, which are smooth manifolds that are expanded by the dynamics, equipped with sufficiently regular probability measures. The coupling of standard pairs was then used to establish statistical properties for two dimensional hyperbolic systems with singularities. In particular Chernov used this technique to reprove many results on two dimensional dispersing billiards [8]. The power and also the flexibility of this method was demonstrated in [9]. The present paper generalizes the technique to high dimensional, piecewise smooth systems with an aim that later this will be a useful tool in the study of physically relevant, complex models.

2 Basic assumptions and consequences

As a first step of the generalization of the coupling method we consider the simplest possible case of smooth, uniformly hyperbolic maps with singularities. We use the assumptions of Chernov from [6], and for the readers convenience we include them here.

Let $\mathcal{M}$ be an open connected domain in a d-dimensional C^∞ Riemannian manifold, such that $\bar{\mathcal{M}}$ is compact. Further let $\xi_1^s = \{\mathcal{M}_1, \dots, \mathcal{M}_N\}$ be a finite collection of disjoint open subsets of $\mathcal{M}$ such that $\bar{\mathcal{M}} = \cup_{i=1}^N \bar{\mathcal{M}}_i$. For any point $x \in \cup\xi_1^s$ we denote by $\xi_1^s(x)$ the unique element of ξ_1^s containing x. We consider a map $F : \cup\xi_1^s \to \mathcal{M}$ such that

(A1) for every $x \in \cup\xi_1^s$, F is a C^2 diffeomorphism of $\xi_1^s(x)$ onto its image. We also assume that F and F^{-1} are twice differentiable (or just $C^{1+\alpha}$) up to the boundaries of their domains (only one-sided derivatives are required at the boundary).

We will refer to the elements of ξ_1^s as the smoothness components of F. For $n \geq 1$ denote by ξ_n^s the measurable partition that consists of the smoothness components of F^n, i.e. for a fixed n the points x and y are in the same element of ξ_n^s iff $\forall i = 0, \dots, n-1$ the points $F^i x$ and $F^i y$ are in the same element of ξ_1^s. Let $\xi_1^u := \{F(\mathcal{M}_1), \dots, F(\mathcal{M}_N)\}$ and for $n > 1$ define ξ_n^u to be the partition that consists of the smoothness components of F^{-n}, in an analogous way as before.

We assume that $\mathcal{S} := \partial\cup\xi_1^s$ is a finite union of smooth compact submanifolds of codimension one, possibly with boundary. We denote by $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r$ the smooth components of $\mathcal{S}$. The set $\mathcal{S}$ will be referred to as the singularity set for

F and in general for $n \geq 1$ denote by $\mathcal{S}^{(n)} := \partial \cup \xi_n^s$ the singularity set for F^n.

We denote by $d(.,.)$ the Riemannian metric in $\mathcal{M}$ and by $m(.)$ the Lebesgue measure (volume) in $\mathcal{M}$. Our additional assumptions on F are as follows.

(A2) F is uniformly hyperbolic, i.e. there exist two families of cones C_x^u and C_x^s in the tangent spaces $T_x\mathcal{M}$, $x \in \bar{\mathcal{M}}$, such that $DF(C_x^u) \subset C_{Fx}^u$ and $DF(C_x^s) \supset C_{Fx}^s$ whenever DF exists, and

$$|DF(v)| \geq \Lambda |v| \quad \forall v \in C_x^u$$

$$|DF^{-1}(v)| \geq \Lambda |v| \quad \forall v \in C_x^s$$

with some constant $\Lambda > 1$. These families of cones are continuous on $\bar{\mathcal{M}}$ and the angle between C_x^u and C_x^s has a uniform positive lower bound.

Technically, the families of cones $C_x^{u,s}$ are specified by two continuous families of linear subspaces $P_x^{u,s} \subset T_x\mathcal{M}$ such that $P_x^u \oplus P_x^s = T_x\mathcal{M}$, and two continuous functions $\alpha^{u,s}(x) > 0$. The cones $C_x^{u,s}$ are defined by

$$\angle(v, P_x^{u,s}) := \min_{w \in P_x^{u,s}} \angle(v,w) \leq \alpha^{u,s}(x) \quad \forall v \in C_x^{u,s}$$

We denote $d_{u,s} = \dim P_x^{u,s}$ (these are independent of x, since $P_x^{u,s}$ are continuous and $\mathcal{M}$ is connected, and $d_u + d_s = d = \dim \mathcal{M}$). The angle between the cones C_x^u and C_x^s is set to $\min\{\angle(v,w) : v \in C_x^u, w \in C_x^s\}$ and we introduce a global constant $C_t > 0$ defined by

$$C_t := \sin(\min_{x \in \bar{\mathcal{M}}} \min_{v,w} \{\angle(v,w) : v \in C_x^u, w \in C_x^s\}) \tag{2.1}$$

For any submanifold $W \subset \mathcal{M}$ we denote by d_W the metric on W induced by the Riemannian metric in $\mathcal{M}$, and by m_W the *normalized* Lebesgue measure on W generated by d_W. For $x \in W$ we will denote by $B_W(x,r)$ the subset of W (if it exists) which is the ball centered at x with radius r in the inner metric d_W. We call U a u-manifold if it is a smooth d_u-dimensional submanifold in $\mathcal{M}$ of finite diameter (in the inner metric d_U) and at every $x \in U$ the tangent space T_xU lies in C_x^u.

(A3) The angle between $\mathcal{S}$ and C^u has a positive lower bound.

Technically, the angle between $\mathcal{S}$ and C_x^u at $x \in \mathcal{S}$ is defined to be $\max\{0, \angle(P_x^u, T_x\mathcal{S}) - \alpha^u(x)\}$. Here $\angle(P_x^u, T_x\mathcal{S}) = \max_{v \in P_x^u} \min_{w \in T_x\mathcal{S}} \angle(v,w)$.
As a consequence of (A3), any u-manifold intersects $\mathcal{S}$ transversally, and the angle between them has a positive lower bound.
It is convenient to assume that for every $\mathcal{S}_i \subset \Gamma$ we have $\partial\mathcal{S}_i \subset \cup_{j \neq i} \operatorname{int} \mathcal{S}_j \cup \partial\mathcal{M}$, i.e. every interior singularity manifold with boundary terminates on some other singularity manifolds or on $\partial\mathcal{M}$. This is not a restrictive assumption, since if this is not the case for some $\mathcal{S}_i \subset \Gamma$, we can extend $\mathcal{S}_i$ until it terminates on other hypersurfaces of $\mathcal{S}$ or on the boundary of $\mathcal{M}$.

A point x of the singularity set $\mathcal{S}^{(m)} = \bar{\mathcal{M}} \setminus \cup \xi_m^s$ of F^m is said to be multiple if it belongs to the boundary of $l \geq 2$ elements of ξ_m^s, and then l is called the multiplicity of x in $\mathcal{S}^{(m)}$.

(A4) There are $K_0 \geq 1$ and $m \geq 1$ such that the multiplicity of any point $x \in \bar{\mathcal{M}} \setminus \cup \xi_m^s$ does not exceed K_0, and $K_0 < \Lambda^m - 1$.

This is a standard assumption which ensures that the singularity manifolds of F^m do not pile up too fast anywhere as m grows. The expansion of any u-manifold U under F^m is hereby guaranteed to be stronger than the cutting (shredding) of U inflicted by the singularities of F^m. It is also standard to assume that $m = 1$ here, which we do, since we can simply consider F^m instead of F. (The assumptions (A1)-(A3) obviously hold for all F^m, $m \geq 1$.)
It is proved in [6, Theorem 1.1] that under the assumptions **(A1) - (A4)** the map F admits a Sinai-Ruelle-Bowen (SRB) measure μ, and any SRB measure μ has a finite number of ergodic components, on each of which it is, up to a finite cycle, mixing and Bernoulli. In view of this, our last additional assumption is quite mild.

(A5) The map F admits a mixing, invariant, SRB-measure μ.

For any $x \in \cup\xi^s_\infty$ and $y \in \cup\xi^u_\infty$ we set

$$E^s_x = \cap_{n\geq 0} DF^{-n}(C^s_{F^n x}), \qquad E^u_y = \cap_{n\geq 0} DF^n(C^u_{F^{-n}y})$$

respectively. It is standard in the literature that

1. E^s_x, E^u_x are linear subspaces in $T_x\mathcal{M}$, $\dim E^{u,s}_x = d_{u,s}$, and $E^s_x \oplus E^u_x = T_x\mathcal{M}$ for $x \in (\cup\xi^u_\infty) \cap (\cup\xi^s_\infty)$,
2. $DF(E^{u,s}_x) = E^{u,s}_{Fx}$, and DF expands vectors in E^u_x and contracts vectors in E^s_x,
3. the subspaces E^u_x and E^s_x are continuous in x (on $\cup\xi^u_\infty$ and $\cup\xi^s_\infty$, respectively), and the angle between them on $(\cup\xi^u_\infty)\cap(\cup\xi^s_\infty)$ has a positive lower bound.

As a consequence, there can be no zero Lyapunov exponents on $(\cup\xi^u_\infty)\bigcap(\cup\xi^s_\infty)$. The space E^u_x is spanned by all vectors with positive Lyapunov exponents, and E^s_x by those with negative Lyapunov exponents. We call a submanifold $W^u \subset \mathcal{M}$ a local unstable manifold (LUM), if F^{-n} is defined and smooth on W^u for all $n \geq 0$, and $\forall x, y \in W^u$ we have $d(F^{-n}x, F^{-n}y) \to 0$ as $n \to \infty$ exponentially fast. Similarly, local stable manifolds (LSM), W^s, are defined. Obviously, $\dim W^{u,s} = d_{u,s}$, and at any point $x \in W^{u,s}$ the tangent space $T_x W^{u,s}$ coincides with $E^{u,s}_x$. We denote by $W^u(x)$, $W^s(x)$ local unstable and stable manifolds containing x, respectively.
We introduce some more notation and recall basic consequences of our assumptions. Let U be a u-manifold. We denote by $\operatorname{diam} U$ the diameter of U in the d_U metric. For any point $x \in U \setminus \mathcal{S}$ denote by $(\mathcal{J}_U F)(x) = |\det(DF|T_x U)|$ the jacobian of the map F restricted to U at x, i.e. the factor of volume expansion on U at the point x (similar notations will be used later for functions also and not just for the map F). For $n \geq 1$ the connected components of $F^n(U \cap \xi^s_n)$ are called components of $F^n U$. Under our assumptions the following results are well known in the literature.

- *Curvature.* We say that the curvature of a u-manifold W is bounded by B if for all $x, y \in W$ we have

$$d_G(T_x W, T_y W) \leq B d(x, y), \tag{2.2}$$

where d_G denotes the distance in the Grassmannian bundle $GR(d_u, T\mathcal{M})$ generated by the Riemannian metric. Then $\exists B > B' > 0$ such that if the curvature of a u-manifold W is at most B', then all the components of F^nW, $n \geq 1$, have curvature at most B. As a result the curvature of any LUM W^u is bounded above by B. We will always assume that the curvature of our u-manifolds is bounded above by B. As a consequence on the microscopic scale each of our u-manifolds looks like a subset of a d_u-dimensional flat hyperplane and the closer we zoom in the more accurate this approach is independently of the u-manifold itself.

- *Distortion bound.* Let $x, y \in U \cap \xi^s_{n-1}$ and $F^n x, F^n y$ belong in one component of F^nU, denote it by V. Then

$$\log \prod_{i=0}^{n-1} \frac{(\mathcal{J}_{F^iU}F)(F^ix)}{(\mathcal{J}_{F^iU}F)(F^iy)} \leq C' d_V(F^nx, F^ny) \tag{2.3}$$

  with some $C' = C'(F) > 0$.

- *Absolute continuity.* Let U_1, U_2 be two sufficiently small u-manifolds, so that any local stable manifold W^s intersects each of U_1 and U_2 in at most one point. Let $U_1' = \{x \in U_1 : W^s(x) \cap U_2 \neq \emptyset\}$. Then we define a map $h : U_1' \to U_2$ by sliding along stable manifolds. This map is often called the **holonomy map**. It is absolutely continuous with respect to the (non normalized) Lebesgue measures ν_{U_1} and ν_{U_2}, and its jacobian (at any point of density of U_1') is bounded, i.e.

$$\frac{1}{C''} \leq \frac{\nu_{U_2}(h(U_1'))}{\nu_{U_1}(U_1')} \leq C''$$

  with some $C'' = C''(F) > 0$.

To verify statistical properties, for example establish bounds on the decay of correlations one has to restrict the class of observables to functions with certain regularity. Usually the class of Hölder continuous functions is an appropriate choice, though a more general class of observables is much more natural in the case of piecewise smooth hyperbolic systems with singularities. This class (which we will define below) was first proposed by Young [15].

Definition 2.1. *For any pair of points $x, y \in \mathcal{M}$ we define their future and past separation time as*

$$s_+(x,y) := \min\{n \geq 0 : y \notin \xi^s_n(x)\} \qquad s_-(x,y) := \min\{n \geq 0 : y \notin \xi^u_n(x)\}$$

These are the first times, when the images $F^n(x)$ and $F^n(y)$ or $F^{-n}(x)$ and $F^{-n}(y)$ respectively lie in different connected components of $\mathcal{M}\setminus\mathcal{S}$ or $\mathcal{M}\setminus\mathcal{S}^{(-1)}$.

Remark 2.2. *As a consequence of uniform hyperbolicity, the compactness of $\bar{\mathcal{M}}$ and the uniform transversality of unstable manifolds and singularities, if x and y lie on one u-manifold $W^u \subset \mathcal{M}$, then*

$$d_{W^u}(x,y) \leq C\Lambda^{-s_+(x,y)} \tag{2.4}$$

and similarly if x and y lie on one s-manifold $W^s \subset \mathcal{M}$, then

$$d_{W^s}(x,y) \leq C\Lambda^{-s_-(x,y)}$$

where C is a constant depending only on the dynamics.

Now we are ready to define our class of observables.

Definition 2.3. *Denote by $\mathcal{H}^+$ the set of all functions $f : \mathcal{M} \to \mathbb{R}$ satisfying, that for any x and y lying on one u-manifold*

$$|f(x) - f(y)| \leq K_f \theta_f^{s_+(x,y)}$$

for some constants $K_f > 0$ and $\theta_f \in (0,1)$.
Similarly $\mathcal{H}^-$ denotes the set of all functions $f : \mathcal{M} \to \mathbb{R}$, such that for any x and y lying on one s-manifold

$$|f(x) - f(y)| \leq K_f \theta_f^{s_-(x,y)}$$

for some constants $K_f > 0$ and $\theta_f \in (0,1)$.
*We call a function f **dynamically Hölder continuous** if $f \in \mathcal{H} := \mathcal{H}^+ \cap \mathcal{H}^-$. We will refer to the constant K_f as the **dynamical Hölder continuity constant** of f and to θ_f as the **dynamical Hölder continuity rate** of f.*

Observe that according to Remark 2.2 the class of ordinary Hölder continuous functions is contained in the class $\mathcal{H}$. Moreover if f is merely piecewise Hölder continuous with sigularities which coincide with some of the singularities of the dynamics, then again $f \in \mathcal{H}$.
One particular motivation of using this class of observables is that it matches with the regularity of a key object of the dynamics, namely the holonomy map. We will discuss this in section 4 in all details.
In the rest of the present section we define the high dimensional generalization of the key object of the coupling procedure. Following the basic idea of Dolgopyat these are u-manifolds with sufficiently regular measures on them.

Definition 2.4. *Let W be an admissible u-manifold equipped with a probability measure ν, which is absolutely continuous with respect to the Lebesgue measure on W. We call (W, ν) a **standard pair** if the density of ν, denoted by ρ is regular, meaning that*

$$|\ln \rho(x) - \ln \rho(y)| \leq C_r \theta_h^{s_+(x,y)},$$

where $C_r > 0$ is a sufficiently large (but fixed) constant and $\theta_h \in (0,1)$ is the dynamical Hölder continuity rate of the holonomy map (to be exactly calculated in Proposition 4.7).

Observe that the regularity of ρ remains the same if we multiply it by a constant. Hence for a standard pair (W, ν) any subcurve $W' \subset W$ with the conditional measure induced by ν on it will itself be a standard pair. In section 3 we will show that a standard pair is mapped by the map F into a collection of standard pairs. As a consequence a class of standard pairs is invariant under the action of F in this sense, which motivates the introduction of a more general object.

Definition 2.5. *A collection of standard pairs $\mathcal{G} = \{(W_i, \nu_i)\}$, $i \in \mathcal{I}$ (where $|\mathcal{I}|$ may be even uncountable) with a probability factor measure $\lambda_\mathcal{G}$ on the index set $\mathcal{I}$ is called a **standard family**. Such a family induces a probability measure $\mu_\mathcal{G}$ on the union $\cup_i W_i$ (and thus on $\mathcal{M}$) defined by*

$$\mu_\mathcal{G}(A) = \int \nu_i(A \cap W_i)\, d\lambda_\mathcal{G}(i) \qquad \forall A \subset \mathcal{M}$$

At this point we are able to state the main results of the paper. We again emphasize that it is not the results that are new, but the method we use to prove them. The notion that a standard family is proper will be defined in section 3, but at the moment it is enough for the reader to imagine a collection of standard pairs with large base u-manifolds.

Theorem 2.6 (Equidistribution)**.** *Let $\mathcal{G}$ be a proper standard family. For any dynamically Hölder continuous function $f \in \mathcal{H}$ and $n \geq 0$*

$$\left| \int_{\mathcal{M}} f \circ F^n \, d\mu_{\mathcal{G}} - \int_{\mathcal{M}} f \, d\mu \right| \leq B_f \Theta_f^n,$$

where $B_f = 2C_\Gamma(K_f + \|f\|_\infty)$ and $\Theta_f = (\max\{\theta_\Gamma, \theta_f\})^{1/2} < 1$.

Theorem 2.7 (Exponential decay of correlations)**.** *Let f and g be dynamically Hölder continuous observables. Then for any $n \geq 0$*

$$\left| \int_{\mathcal{M}} f \cdot (g \circ F^n) \, d\mu - \int_{\mathcal{M}} f \, d\mu \cdot \int_{\mathcal{M}} g \, d\mu \right| \leq B_{f,g} \Theta_{f,g}^n,$$

where

$$\Theta_{f,g} = (\max\{\sqrt{\theta_\Gamma}, \theta_f, \sqrt{\theta_g}\})^{1/2} < 1,$$

with $\theta_\Gamma \in (0,1)$ to be defined later in the paper and

$$B_{f,g} = 4C_\Gamma C_p(K_f \|g\|_\infty + K_g \|f\|_\infty + \|f\|_\infty \|g\|_\infty),$$

with global constants $C_\Gamma > 0$ and $C_p > 0$ also defined later.

The proof of these theorems will be given in section 5.

3 Regularity properties

In this section we collect some of the frequently used constructions and well known results from the literature, which we will need later in our arguments. We follow the work of Chernov [6] (sometimes even word by word) and for proofs and more details we suggest the reader to have a look at the original paper.
Our assumption (A4) implies that $\exists \delta' > 0$ such that any δ'-ball in $\mathcal{M}$ intersects at most K_0 smooth components of $\mathcal{S}$. We choose a parameter δ_0 to be much smaller than δ' and the minimum radius of curvature of singularity manifolds $\mathcal{S}_i \subset \mathcal{S}$.

Definition 3.1. *A connected u-manifold U is* ***admissible*** *if*

- *its curvature is at most B everywhere,*
- $\operatorname{diam} U \leq \delta_0$*,*
- *its boundary ∂U is piecewise smooth, i.e. it is a finite union of smooth compact submanifolds of dimension $d_u - 1$, possibly with boundary.*

*Using that admissible u-manifolds always have bounded curvature we assume that our δ_0 is so small that the tangent spaces at different points of these manifolds are almost parallel. Let U be an admissable u-manifold and $V \subset U$ an open subset with piecewise smooth boundary. $\forall x \in V$ denote by $V(x)$ the connected component of V that contains x. We say that V is n-**admissible** for some $n \geq 0$, if F^n is smooth on V and $\forall x \in V$ the u-manifold $F^n V(x)$ is admissible. Finally we define a function $r_{V,n}(x)$ on V by*

$$r_{V,n}(x) = d_{F^n V(x)}(F^n x, \partial F^n V(x))$$

It will be important to control the size of u-manifolds (at least on the average) when iterating them forward by F. This was done previously in the literature. Here we just recall the necessary definitions and results. For the proofs the reader should take a look at [6] and/or [7].
For any $\delta > 0$ denote by $\mathcal{U}_\delta$ the δ-neighborhood of the closed set $\mathcal{S}$.

Definition 3.2. *Let $\delta > 0$ and W be an admissable u-manifold. Two sequences of open subsets $W = W_0^1 \supset W_1^1 \supset W_2^1 \supset \dots$ and $W_n^0 \subset W_n^1 \setminus W_{n+1}^1$, $n \geq 0$, are said to make a δ-**filtration** of W if $\forall n \geq 0$*

- *the sets W_n^1 and W_n^0 are n-admissable subsets of W,*
- $m_W(W_n^1 \setminus (W_{n+1}^1 \cup W_n^0)) = 0$,
- $F^n W_{n+1}^1 \cap \mathcal{U}_{\delta\Lambda^{-n}} = \emptyset$ *and* $F^n W_n^0 \subset \mathcal{U}_{\delta\Lambda^{-n}}$.

We put $w_n^1 = m_W(W_n^1)$, $w_n^0 = m_W(W_n^0)$ and $W_\infty^1 = \cap_{n\geq 0} W_n^1$.

Remark 3.3. *In [6] this was called a refined u-filtration and the $\delta \to 0$ limit case gives back the so called admissible u-filtration of W. We will refer to this later one as the 0-filtration of W. The reader should imagine a δ-filtration of W as follows. Take the n-th image $F^n(W)$ and consider those points that lie closer to a singularity than $\delta\Lambda^{-n}$ exactly at this iterate (i.e. not for any $i < n$). Basically the preimage of these points form the set W_n^0. Then remove the union of the closures of W_i^0's, $i = 0, \dots, n$, from W and dice the remaining set with a specially (i.e. in some sense optimally) chosen $\delta_0/\sqrt{d_u}$-grid of $d_u - 1$ dimensional hyperplanes. In this way we ensure that the diameter of the connected components in $F^n(W_n^1)$ will not increase δ_0, so they remain admissible u-manifolds. Note also that for a δ-filtration, a stable disk $W_\delta^s(x)$ of radius δ exists at every point $x \in W_\infty^1$.*

The following is proved in [6], see the proof of Theorem 2.1 and Theorem 4.1.

Theorem 3.4. *Let W be an admissable u-manifold and $\delta > 0$. There are constants $\alpha \in (0, 1)$ and $\beta, C' > 0$ and there is a δ-filtration of W such that*

1. $\forall n \geq 1$ *and* $\forall \varepsilon > 0$ *we have*
$$m_W(r_{W_n^1,n} < \varepsilon) \leq (\alpha\Lambda)^n \cdot m_W(r_{W,0} < \varepsilon/\Lambda^n) + \varepsilon\beta\delta_0^{-1}(1 + \alpha + \dots + \alpha^{n-1}) m_W(W)$$
2. $\forall n \geq 0$ *and* $\varepsilon > 0$
$$m_W(r_{W_n^0,n} < \varepsilon) \leq (3K_0 + 1) \cdot m_W(r_{W_n^1,n} < \varepsilon)$$

3. and

$$m_W(W_n^0) \le m_W(r_{W_n^0,n} < C'\delta\Lambda^{-n})$$

These results are often formulated in a somewhat weaker form (just as in the given references) with the help of the following function.

Definition 3.5. *Let W be an admissible u-manifold and $V \subset W$ an n-admissible open subset. We set*

$$Z[W,V,n] := \sup_{\varepsilon>0} \frac{m_W(x \in V : r_{V,n}(x) < \varepsilon)}{\varepsilon \cdot m_W(W)}$$

A brief description of the geometrical meaning and a certain characterization of the value of the Z-function can be found in [6, 7].
We also recall basic consequences of Theorem 3.4 that already appeared in [6, Corollary 4.3]. For a δ_2-filtration of an admissible u-manifold W (δ_2 to be specified later) we introduce the following notations:

$$Z_n^1 = Z[W, W_n^1, n] \qquad Z_n^0 = Z[W, W_n^0, n]$$

Corollary 3.6. *Let $\bar\beta = 2\beta/(1-\alpha)$, $\delta_1 \le \delta_0/(2\bar\beta)$, $a = -(\ln\alpha)^{-1}$ and $b = \max\{0, a\ln[\delta_0(1-\alpha)/\beta]\}$. Further let $\bar Z_0 = \max\{Z_0, \bar\beta/\delta_0\}$. Then*

1. $Z_n^1 \le \bar Z_0$ *and* $Z_n^0 \le (3K_0+1)\bar Z_0$ *for all* $n \ge 0$,
2. $Z_n^1 \le \bar\beta/\delta_0 = (2\delta_1)^{-1}$ *for all* $n \ge a\ln Z_0 + b$,
3. $w_n^0 \le C''\bar Z_0\delta_2\Lambda^{-n}$ *for all* $n \ge 0$, *where* $C'' = (3K_0+1)C'$,
4. $w_n^1 \ge 1 - C''\bar Z_0\delta_2/(1-\Lambda^{-1})$ *for all* $n \ge 1$.

With the help of these results one can construct special sets (called rectangles in the literature), which we will also need in our later arguments. These are products of two Cantor-like sets. We include the steps of the construction here, as it is done in [7, Section 4].
Choose our parameter δ_0 and hence δ_1 (along with the previous restrictions) to be so small that $\exists x_0 \in \mathcal{M}$ such that the local unstable manifold $W^u_{\delta_1}(x_0)$ (a d_u-dimensional disk around x_0 with radius δ_1 is the inner metric of the manifold) exists. Consider its "central part" $W^u_{\delta_1/3}(x_0)$ (which is a perfect ball in its own metric) and note that according to its bounded curvature (and the sufficiently small value of δ_1) we have

$$Z[W^u_{\delta_1/3}(x_0), W^u_{\delta_1/3}(x_0), 0] \le 4d_u/\delta_1 \tag{3.1}$$

Now we set the value of δ_2 by the relation

$$\frac{\delta_2}{\delta_1} = \frac{1-\Lambda^{-1}}{40C''d_u} \tag{3.2}$$

From the previous observation on the Z-function and from part 4 of Corollary 3.6 it follows that there is a δ_2-filtration of $W^u_{\delta_1/3}(x_0)$ such that $m_{W^u_{\delta_1/3}(x_0)}(W^1_\infty) \ge 0.9 \cdot m_{W^u_{\delta_1/3}(x_0)}(W^u_{\delta_1/3}(x_0))$. This means that for, at least, 90% of the points $x \in W^u_{\delta_1/3}(x_0)$ (with respect to the Lebesgue measure on $W^u_{\delta_1/3}(x_0)$) the stable disk $W^s_{\delta_2}(x)$ exists. Now we basically consider all intersections of these stable disks and unstable manifolds that are close to $W^u_{\delta_1/3}(x_0)$ in a sense we describe below (as in [7, Section 4]).

Let $x \in \mathcal{M}$ and $r \in (0, \delta_0)$. We denote by $S_r(x)$ any s-manifold that is a ball of radius r around x in its own metric (note that this is not a unique object) and call it an s-disk. In order to define s-disks also around points close to $\partial\mathcal{M}$ we extend the cone field C^s continuously beyond $\partial\mathcal{M}$ into the δ_0-neighborhood of $\mathcal{M}$.

Let W be an admissible u-manifold and $x \in \mathcal{M}$. Since δ_0 is small enough any s-disk $S_{\delta_0}(x)$ can meet W in at most one point. We call

$$H_x(W) = \{y \in W | y = S_{\delta_0}(x) \cap W \text{ for some } S_{\delta_0}(x)\}$$

the s-shadow of x on W. We say that a point $x \in \mathcal{M}$ is overshadowed by the u-manifold W if $\forall S_{\delta_0}(x)$ we have $S_{\delta_0}(x) \cap W \neq \emptyset$. In this case, of course, $d(x, W) \leq \delta_0$. We call

$$d^s(x, W) = \sup_{S_{\delta_0}(x)} d_{S_{\delta_0}(x)}(x, S_{\delta_0}(x) \cap W)$$

the s-distance from x to W. Let W and W' be two admissible u-manifolds. We call

$$H_W(W') = \cup_{x \in W} H_x(W')$$

the s-shadow of W on W'. We say that W' overshadows W if it overshadows every point of W. In this case we define

$$d^s(W, W') = \sup_{x \in W} d^s(x, W')$$

the s-distance from W to W'. It is not symmetric because no two u-manifolds can simultaneously overshadow each other. Geometrically W' overshadows W if W is close to W' and W' stretches all the way along W and a little beyond it. We go into details in the next lemma.

Lemma 3.7. *Let x and y be two closeby points in the phase space ($d(x, y) \ll \delta_0$) and let W_1 and W_2 be two u-manifolds containing x and y respectively. We claim two things:*

1. *for a fixed $R > 0$ the ball $B_1(x, \rho) \subseteq W_1$ overshadows the ball $B_2(y, R) \subseteq W_2$ as long as $\rho \geq \frac{2}{C_t}(R + d(x, y))$,*
2. *in addition if originally for some $r < R$ the ball $B_1(x, r) \subseteq W_1$ was overshadowed by $B_2(y, R) \subseteq W_2$ with s-distance $d^s(B_1(x, r), B_2(y, R)) \leq \delta$ (with some small $\delta > 0$), then $d^s(B_2(y, R), B_1(x, \rho)) \leq 2(\frac{2R}{C_t r} + 1)\delta$,*

where C_t is the constant defined by (2.1).

Proof. Proof of claim 1.

Since both families of cones are continuous on $\bar{\mathcal{M}}$ it is clear that if x and y are close to each other, then the cones C^u_x and C^s_y are still transversal and the sine of the minimum angle between them is just slightly different from C_t. The uniform curvature bound (2.2) implies that it is actually enough to prove the statement for u-manifolds that are subsets of d_u-dimensional hyperplanes in $\mathbb{R}^d$.

So let T_yW_2 and T_xW_1 be d_u-dimensional affine hyperplanes containing W_2 and W_1 respectively. Choose a point $y' \in T_yW_2$ such that $d(y,y') \le R$ and let $T_{y'}S$ be a d_s-dimensional affine hyperplane through y' contained in the stable cone $C^s_{y'}$. Consider the parallel translation of T_xW_1 at y' and denote it by $T_{y'}W_1$. By transversality of the cones C^u_x and $C^s_{y'}$ we have that $T_{y'}W_1 \oplus T_{y'}S = T_{y'}\mathcal{M}$ and hence $T_{y'}S$ will have a unique intersection with T_xW_1 we denote it by z. Now we have to estimate $d(x,z)$. To this end we introduce the vectors $b = y - x$ and $v = y' - y$ and denote their decompositions in $T_{y'}W_1 \oplus T_{y'}S$ by $b^u + b^s$ and $v^u + v^s$ respectively. Observe that then $d(x,z) \le \|v^u\| + \|b^u\|$. These norms can be estimated in the same way, for example for $\|v^u\|$ choose a unit vector $v^s_\perp$ in the plane spanned by v^u and v^s, which is orthogonal to v^s. Then calculating the scalar product of this and v leads to $\langle v, v^s_\perp\rangle = \langle v^u, v^s_\perp\rangle$ and hence

$$\|v^u\| = \frac{\|v\| \cdot \cos\angle(v, v^s_\perp)}{\cos\angle(v^u, v^s_\perp)}.$$

Here $\|v\|$ is of course at most R and the denominator can be estimated from below by C_t, so $\|v^u\| \le \frac{R}{C_t}$. Similar estimate holds for the norm $\|b^u\|$ except that in that case $\|b\| = d(x,y)$, so $\|b^u\| \le \frac{d(x,y)}{C_t}$. Therefore the combination of the two bounds gives that

$$d(x,z) \le \|v^u\| + \|b^u\| \le \frac{1}{C_t}(R + d(x,y)).$$

Since s- and u-manifolds actually have some bounded curvature we have to somewhat increase the value of our estimate, but $\frac{2}{C_t}(r + d(x,y))$ will be enough provided that our δ_0 is sufficiently small.

Proof of claim 2.

We continue using the notations introduced above and the flatness assumption on the u-manifolds. We want to give an upper bound on $d(y',z)$ using the shadowing assumption we made. Denote by e the (1-dimensional) line connecting x and z and let $e \cap \partial B_1(x,r) = \{m_1, m_2\}$, where m_1 is further away from z than m_2. Consider the parallel translation of $T_{y'}S$ at m_1. Due to our assumption on shadowing we know that the intersection $p_1 := T_{m_1}S \cap B_2(y,R)$ exists and $\|p_1 - m_1\| \le \delta$. We introduce $q_1 := m_2 + (p_1 - m_1)$ and also $p_2 := T_{m_2}S \cap B_2(y,R)$. It is evident that $\|q_1 - p_1\| = 2r$ and by our knowledge on the s-distance $\|q_1 - p_2\| \le \|q_1 - m_2\| + \|m_2 - p_2\| \le 2\delta$. From these it follows that

$$\frac{\sin\angle(p_2 - p_1, q_1 - p_1)}{\sin\angle(p_1 - p_2, q_1 - p_2)} = \frac{\|p_2 - q_1\|}{\|p_1 - q_1\|} \le \frac{\delta}{r} \tag{3.3}$$

and hence $\sin\angle(p_2 - p_1, q_1 - p_1) \le \frac{\delta}{r}$.

The final (we promise!) point to introduce is $q_2 := z + p_1 - m_1$. It is evident that p_1, q_1 and q_2 are on the same 1-dimensional line, namely on the line e translated by the vector $p_1 - m_1$. From this it also follows that p_1, p_2 and y' are collinear too and hence the angles $\angle(y' - p_1, q_2 - p_1)$ and $\angle(p_2 - p_1, q_1 - p_1)$ are the same. The estimate goes on as $\|y' - z\| \le \|y' - q_2\| + \|q_2 - z\|$, where $\|q_2 - z\| = \|m_1 - p_1\| \le \delta$ as we saw earlier and for the first term we use our observation on the angles to conclude that

$$\frac{\|y' - q_2\|}{\|y' - p_1\|} = \frac{\sin\angle(p_2 - p_1, q_1 - p_1)}{\sin\angle(p_1 - y', q_2 - y')}.$$

Then using that $\|y' - p_1\| \leq 2R$, the estimate (3.3) and the uniform transversality of stable and unstable cones we get that

$$\|y' - q_2\| \leq \frac{2R\delta}{C_t r},$$

which then implies $\|y' - z\| \leq (\frac{2R}{C_t r} + 1)\delta$. Since s- and u-manifolds actually have some bounded curvature we have to somewhat increase the value of our estimate, but $2(\frac{2R}{C_t r} + 1)\delta$ will be enough provided that our δ_0 is sufficiently small. $\square$

To have uniform control on the size of a rectangle in the stable direction we choose a small parameter δ_3 as

$$\delta_3 \leq \min\{c_s\delta_2, \delta_2/3, \delta_1/120\}, \tag{3.4}$$

where c_s is a constant depending on Λ and C_t (we will make this dependence explicit in subsection 5.3).

!!!Inner Comment!!! 3.8. *A téglák definíciójának módosítása esetén felül kell vizsgálni c_s értékét!*

We remark that $\delta_3 \leq \delta_2/3$ together with (3.2) actually implies that $\delta_3 \leq \delta_1/120$ so we just included this here for convenience. After this we give the definition of a rectangle and a corresponding set called a magnet together with notions we will frequently use in the rest of the paper.

!!!Inner Comment!!! 3.9. *Ha az átmetszési lemma bizonyítását az új megközelítés szerint csináljuk (Lai-Sang) érvelését követve, akkor innen kihagyható egy-két dolog. Nincs szükség például a bezoomolós szövegre, vagyis fölösleges bevezetni az r_w paramétereket és az ezzel kapcsolatos plusz követelményeket.*

Definition 3.10. *For an admissible u-manifold W the magnet σ_W and the adapted rectangle R_W are defined through the following steps. In our applications it will always be assumed that W contains a ball $B_W(x, R)$ of radius $R \geq \delta_1$ for some $x \in W$. We will use the expression that σ_W and R_W are **built on** the u-manifold W. Take $B_W(x, \delta_1/3)$, the central part of the mentioned ball, and consider a δ_2-filtration of it. This results in a set W^1_∞, such that $\forall y \in W^1_\infty$ the stable disk $W^s_{\delta_2}(y)$ exists and the Lebesgue measure of W^1_∞ is at least 0.9 times the Lebesgue measure of $B_W(x, \delta_1/3)$ (as we described after (3.2)). We use the Lebesgue density theorem to choose a density point y_0 of the set W^1_∞ and a radius r_W such that $B_W(y_0, r_W) \cap W^1_\infty$ has positive Lebesgue measure. We define the magnet σ_W as*

$$\sigma_W = \{W^s_{\delta_2}(x) | x \in W^1_\infty \cap B_W(y_0, r_W)\}$$

*and we will call the ball $B_W(y_0, r_W)$ **the base of** σ_W (and also **the base of** the rectangle R_W yet to be defined). Finally we define the rectangle R_W as follows: $y \in R_W$ iff $y = W^s_{\delta_2}(x) \cap W^u$ for some $x \in W^1_\infty \cap B_W(y_0, r_W)$ and for some local unstable manifold W^u with the following property. The unstable manifold W^u must contain a d_u-dimensional ball $B_{W^u}(z, 4r_W)$ for some point $z \in W^u$ such that the center of it $B_{W^u}(z, 2r_W)$ overshadows the base of the magnet $B_W(y_0, r_W)$ with s-distance $d^s(B_W(y_0, r_W), B_{W^u}(z, 2r_W)) \leq \delta_3$, where*

δ_3 *satisfies* (3.4)*. We will use the expression that such a* W^u *is a* ***building block*** *of* R_W *and we will refer to* r_W *as the* ***unstable-*** *and to* δ_3 *as the* ***stable size*** *of the rectangle* R_W*.*

Remark 3.11. *Note that we have a considerable flexibility in choosing the parameters in the previous definition, in particular we have very little restrictions on the unstable size* r_W*. To have a better understanding of the structure of the rectangle* R_W *we emphasize that local unstable manifolds are unique, so the* W^u*'s do not intersect each other, but some may (and do) intersect* W *since it is only assumed to be a u-manifold. This also shows that points of* W *are not necessarily points of* R_W*.*

Now we turn on investigating the properties of standard pairs and their iterates by the map F. The first important observation is that the class of standard pairs is invariant under F in the following sense:

Proposition 3.12. *Let* (W, ν) *be a standard pair. For each* $n \geq 0$*, denote by* $W_{i,n}$ *the connected components of* $F^n(W)$*. Then* $F^n(\nu) = \sum_i c_{i,n}\nu_{i,n}$*, where* $\sum_i c_{i,n} = 1$ *and each* $(W_{i,n}, \nu_{i,n})$ *is a standard pair.*

Proof. The proof is a slight modification of the proof of [8, Proposition 3.1]. By induction it is enough to prove the statement for $n = 1$. U-manifolds are expanded under the action of F so it may happen that a connected component of the image $F(W)$ is not admissible since its diameter exceeds δ_0 (but this can be the only reason for that). We can artificially chop such components into admissible u-manifolds, this is actually done in the construction of a 0-filtration of W (cf. Remark 3.3 and the lines after that). So what really needs to be proven is that the regularity of the density is kept.
Consider a connected component $W_{i,1}$ of $F(W)$ and let $x, y \in W_{i,1}$. Denote by $\rho_{i,1}$ the density of $\nu_{i,1}$ and by $x_1 = F^{-1}(x)$ and $y_1 = F^{-1}(y)$ the preimages of the points. Then obviously $s_+(x, y) = s_+(x_1, y_1) - 1$. Using the bound (2.3) on distortions and (2.4) one can conclude that

$$
\begin{aligned}
|\ln \rho_{i,1}(x) - \ln \rho_{i,1}(y)| &\leq |\ln \rho(x_1) - \ln \rho(y_1)| + \\
&+ |\ln(\mathcal{J}_W F^{-1})(x) - \ln(\mathcal{J}_W F^{-1})(y)| \leq \\
&\leq C_r \theta_h^{s_+(x_1,y_1)} + C' d(x, y) \leq \\
&\leq C_r \theta_h \theta_h^{s_+(x,y)} + C \theta_h^{s_+(x,y)}
\end{aligned}
\tag{3.5}
$$

for some constant $C > 0$. Thus it is enough to assume that C_r is so large that $C_r\theta_h + C \leq C_r$. □

Remark 3.13. *Note that* (3.5) *describes the process how densities regularize. Imagine that* W *is an admissible u-manifold and* ν *is a probability measure on it with density* $\rho(x)$ *such that* $|\ln \rho(x) - \ln \rho(y)| \leq C_0 \theta_h^{s_+(x,y)}$ *with* $C_0 > C_r$*. That is* (W, ν) *is almost a standard pair, the only issue is that the density is not regular enough. Then an argument similar to* (3.5) *shows that*

$$
|\ln \rho_n(x) - \ln \rho_n(y)| \leq (C_0 \cdot \theta_h^n + C)\theta_h^{s_+(x,y)},
$$

where ρ_n *is the* n*-th image of the density* ρ *and the points* x, y *are assumed to be in a connected component of the image* $F^n(W)$*. Clearly if* n *is large enough* *then* $(C_0 \cdot \theta^n + C) \leq C_r$ *holds and so at that time the image of* (W, ν) *becomes a standard family.*

With the more general definition of standard families (Definition 2.5) Proposition 3.12 says that F^n transforms a standard pair into a finite standard family whose factor measure is defined by the sequence of the coefficients $\{c_{i,n}\}$. Similarly, any standard family $\mathcal{G}$ is mapped by F^n into another standard family $\mathcal{G}_n = F^n(\mathcal{G})$. It is easy to see that $\mu_{\mathcal{G}_n} = F^n(\mu_{\mathcal{G}})$.

Definition 3.14. *For any standard family* $\mathcal{G} = \{(W_i, \nu_i)\}$, $i \in \mathcal{I}$ *we define two functions. For every* $i \in \mathcal{I}$ *and* $x \in W_i$ *we introduce*

$$r_{\mathcal{G}}(x) := d_{W_i}(x, \partial W_i)$$

(cf. our notation introduced in Definition 3.1) which is a function on $\cup_i W_i$*. Using this notation we define the key tool (which is an extension of the Z-function from Definition 3.5) in the control of the sizes of u-manifolds.*

$$\mathcal{Z}_{\mathcal{G}} := \sup_{\varepsilon > 0} \frac{\mu_{\mathcal{G}}(x : r_{\mathcal{G}}(x) < \varepsilon)}{\varepsilon}$$

The value of $\mathcal{Z}_{\mathcal{G}}$ measures the size of the standard family $\mathcal{G}$ in the following sense. If $\mathcal{Z}_{\mathcal{G}} \leq C_0$, then for every $\varepsilon > 0$ the $\mu_{\mathcal{G}}$ measure of those points that lie closer than ε to the boundary $\partial \cup_i W_i$ is at most $C_0 \cdot \varepsilon$. So if ε is small then the measure of those points $x \in \cup_i W_i$ such that the d_u-dimensional ball in $W_i(x)$, with radius ε centered at x is contained in W_i, is large (can be made arbitrarily close to 1 by choosing ε small enough).
The growth of u-manifolds in standard families under the iteration of F is formulated in the following sense, which is a consequence of the first part of Theorem 3.4.

Lemma 3.15. *[Growth lemma] Let* $\mathcal{G} = \{(W_i, \nu_i)\}$, $i \in \mathcal{I}$, *be a standard family and* $\mathcal{G}_n = F^n(\mathcal{G})$*. Then for all* $n \geq 0$ *and* $\varepsilon > 0$ *we have* $\mathcal{Z}_{\mathcal{G}_n} \leq c_1 \alpha^n \mathcal{Z}_{\mathcal{G}} + c_2$ *for some constants* $c_i = c_i(F) > 0$, $i = 1, 2$.

Proof. Using that the measures on standard pairs are uniformly equivalent to Lebesgue, one can adapt the first statement of Theorem 3.4 with 0-filtrations on W_i's and for the measures ν_i. Then integrating with respect to the factor measure and taking the supremum in ε gives the lemma with $c_1 = e^{2C_r\theta_h}$ and $c_2 = \frac{1}{\delta_0}(e^{C_r\theta_h}\beta\frac{1}{1-\alpha})$. □

Corollary 3.16. *For all* $n \geq \chi \ln \mathcal{Z}_{\mathcal{G}}$ *we have* $\mathcal{Z}_{\mathcal{G}_n} \leq c_3$ *for some constants* $\chi, c_3 > 0$.

Now we comment on how $\mathcal{Z}_{\mathcal{G}_n}$ measures the size of the standard family $\mathcal{G}_n$.

Definition 3.17. *We call a standard pair* (W, ν) δ_1***-proper*** *if there is a point* $x \in W$ *such that within* W *there exists a ball around* x *with radius* δ_1.

Corollary 3.16 tells that after a certain number of iterations, depending only on $\mathcal{Z}_{\mathcal{G}}$, i.e. the initial size of the family, the value of $\mathcal{Z}_{\mathcal{G}_n}$ drops below a fixed constant c_3. This means that $\mathcal{G}_n$ will be large in the following sense. If we set the value of our parameter δ_1 as

$$\delta_1 \leq \min\{\delta_0/(2\bar{\beta}), 1/(2c_3)\} \tag{3.6}$$

(so it still satisfies our previous restriction on δ_1 in Corollary 3.6) then for every $n \geq \chi \ln \mathcal{Z}_\mathcal{G}$ the $\mu_{\mathcal{G}_n}$-measure of those points around which exists a ball of radius δ_1 in the u-manifolds from $\mathcal{G}_n$ is at least 1/2. Alternatively the overall $\mu_{\mathcal{G}_n}$-measure of u-manifolds of δ_1-proper standard pairs in $\mathcal{G}_n$ is at least 1/2. To eliminate the dependence on the initial size we introduce the following notion.

Definition 3.18. *A standard family $\mathcal{G}$ is said to be* ***proper*** *if $\mathcal{Z}_\mathcal{G} \leq C_p$, where C_p is a sufficiently large ($C_p > c_3$) fixed constant.*

Remark 3.19. *It follows from the lines of [6, Section 3] that the partition ξ^u_∞ of $\mathcal{M}$ into maximal unstable manifolds with the conditional SRB-measures on them and the factor measure induced by μ forms an F invariant, proper standard family (provided that we choose C_p large enough).*

!!!Inner Comment!!! 3.20. *Az egész holonómiás fejezetben megfontolható a tömörítés, pl. Lemma 4.4 bizonyításában vagy később a Gram-Scmidt-es érvelésnél. Ezek talán túl részletesek, kevesebb indoklás is elég lehet.*

4 Regularity of the holonomy map

Let W_1 and W_2 be sufficiently small u-manifolds close to each other, so that any local stable manifold W^s intersects each of W_1 and W_2 in at most one point and let $W_1' = \{x \in W_1 : W^s(x) \cap W_2 \neq \emptyset\}$. Recall that the **holonomy map** $h : W_1' \to W_2$ is defined by sliding along stable manifolds. In this section we prove some regularity properties of the holonomy map, which we need later for the coupling. We start by recalling a well known result (see [1, 13] for example or [10, Theorem 5.39]).

Proposition 4.1. *The Jacobian of the holonomy map can be expressed as follows.*

$$(\mathcal{J}_{W_1} h)(x) = \lim_{n\to\infty} \frac{(\mathcal{J}_{W_1} F^n)(x)}{(\mathcal{J}_{W_2} F^n)(h(x))} = \lim_{n\to\infty} \prod_{i=0}^{n-1} \frac{(\mathcal{J}_{W_1^i} F)(x_i)}{(\mathcal{J}_{W_2^i} F)(h(x_i))}$$

Here we used the notations $x_i = F^i(x)$ and $W^i_{1,2} = F^i(W_{1,2})$.

We first investigate the tail behaviour of this infinite product. The argument goes through a number of lemmas. Then we will use the result together with further estimates to derive two things: the dependence of the Jacobian of the holonomy map on the initial geometry and that $\ln(\mathcal{J}_{W_1} h)(x)$ is dynamically Hölder continuous on its domain.

Lemma 4.2. *There are constants $C > 0$, $N_0 \geq 0$ and $\theta \in (0, 1)$, such that*

$$\left| \sum_{i=n}^{\infty} \ln(\mathcal{J}_{W_1^i} F)(x_i) - \ln(\mathcal{J}_{W_2^i} F)(h(x_i)) \right| \leq C \cdot \theta^n$$

for all $n \geq N_0$.

Proof. Note that we need to compare the *unstable* Jacobian in two different points of a *stable* manifold. We would like to use the regularity of the total Jacobian and also the distortion bound on the inverse dynamics to establish the desired estimate. To this end we have to find the connection between the total Jacobian and the Jacobian of the dynamics when restricted to u-manifolds or stable manifolds. This is the purpose of the next lemma.

Lemma 4.3. *Let W be a u-manifold and W_s be a stable manifold, both containing the point x. We claim that there is an explicitely computable constant $C(x)$ (that we will calculate), such that $(\mathcal{J}_M F)(x) = C(x) \cdot (\mathcal{J}_W F)(x) \cdot (\mathcal{J}_{W_s} F)(x)$.*

Proof. Let $\{\underline{e}_i(x)\}_{i=1}^{d_u}$ and $\{\underline{f}_j(x)\}_{j=1}^{d_s}$ be orthonormal bases in $T_x W$ and in $E_x^s(= T_x W^s)$ respectively. By arranging these vectors into columns we get the matrices $O_u(x)$ and $O_s(x)$ with dimensions $d \times d_u$ and $d \times d_s$. We iterate these column vectors forward by the tangent map and get $\underline{u}_i = D_x F(\underline{e}_i(x))$ and $\underline{s}_j = D_x F(\underline{f}_j(x))$ forming the matrices U and S in a similar way as before. Now choose again orthonormal bases, but this time in the forward iterates of the previous manifolds, i.e. $\{\underline{e}_i(F(x))\}_{i=1}^{d_u}$ and $\{\underline{f}_j(F(x))\}_{j=1}^{d_s}$ in $T_{F(x)}F(W)$ and in $E^s_{F(x)}$ respectively. Let these vectors be the columns of the matrices $O_u(F(x))$ and $O_s(F(x))$. Then there exist matrices A and B with dimensions $d_u \times d_u$ and $d_s \times d_s$ such that $U^T = A \cdot O_u^T(F(x))$ and $S^T = B \cdot O_s^T(F(x))$. Now on the one hand $(\mathcal{J}_W F)(x) = \sqrt{\det U^T U} = \sqrt{\det AA^T}$ and $(\mathcal{J}_{W_s} F)(x) = \sqrt{\det S^T S} = \sqrt{\det BB^T}$. On the other hand the total Jacobian of F is the volume spanned by the vectors $\underline{u}_i$ and $\underline{s}_j$ divided by the volume spanned by the vectors $\underline{e}_i(x)$ and $\underline{f}_j(x)$. Here the square of the numerator can be calculated as $\det\big((U||S)^T \cdot (U||S)\big)$, where $||$ denotes concatenation. This can be expanded as

$$\det\begin{pmatrix} U^T U & U^T S \\ S^T U & S^T S \end{pmatrix} = \det\begin{pmatrix} AA^T & AO_u^T(F(x))O_s(F(x))B^T \\ BO_s^T(F(x))O_u(F(x))A^T & BB^T \end{pmatrix}$$
$$= \det(AA^T) \cdot \det(BB^T - BO_s^T(F(x))O_u(F(x))A^T(AA^T)^{-1}AO_u^T(F(x))O_s(F(x))B^T)$$
$$= \det(AA^T) \cdot \det(BB^T - BO_s^T(F(x))O_u(F(x))O_u^T(F(x))O_s(F(x))B^T)$$
$$= \det(AA^T) \cdot \det(BB^T) \cdot \det(Id_{d_s} - (O_s^T(F(x))O_u(F(x)))(O_s^T(F(x))O_u(F(x)))^T) \tag{4.1}$$

Observe that the square of the denominator can be calculated in the same way by replacing $F(x)$ with x and also A and B with identity matrices of the corresponding sizes. Let us use the notation $\Psi(x) = (O_s^T(x)O_u(x))(O_s^T(x)O_u(x))^T$. Then the connection between the Jacobians is the following.

$$((\mathcal{J}_M F)(x))^2 = \frac{((\mathcal{J}_W F)(x))^2 \cdot ((\mathcal{J}_{W_s} F)(x))^2 \cdot \det(Id_{d_s} - \Psi(F(x)))}{\det(Id_{d_s} - \Psi(x))}$$

hence

$$C(x) = \sqrt{\frac{\det(Id_{d_s} - \Psi(F(x)))}{\det(Id_{d_s} - \Psi(x))}}$$

□

Now consider the logarithm of one term in Proposition 4.1, which is a difference of the unstable Jacobians along a stable manifold. With the help of the previous lemma we can rewrite it as

$$\ln(\mathcal{J}_{W_1^n}F)(x_n) - \ln(\mathcal{J}_{W_2^n}F)(h(x_n)) = \ln(\mathcal{J}_M F)(x_n) - \ln(\mathcal{J}_M F)(h(x_n)) +$$
$$+\ln(\mathcal{J}_{W_s^n}F)(h(x_n)) - \ln(\mathcal{J}_{W_s^n}F)(x_n) + \frac{1}{2}(\ln\det(Id_{d_s} - \Psi(F(h(x_n)))) - \ln\det(Id_{d_s} - \Psi(F(x_n)))) +$$
$$+ \frac{1}{2}(\ln\det(Id_{d_s} - \Psi(x_n)) - \ln\det(Id_{d_s} - \Psi(h(x_n)))) \quad (4.2)$$

The first difference here in absolute value is less than $const \cdot d(x_n, h(x_n)) = const \cdot \Lambda^{-n}$ because the map F is assumed to be piecewise C^2, so its total jacobian is C^1 and the points x_n and $h(x_n)$ are in the same smoothness component for any $n \geq 0$. For the second difference remember that we will have to sum up these in n. So first do the summation in n from N_0 to ∞ and then apply the distortion bound on the inverse dynamics to conclude that the absolute value of the whole sum of these differences is less than $const \cdot d_{W_s}(x_{N_0}, h(x_{N_0})) = const \cdot \Lambda^{-N_0}$. The remaining terms will require a bit more work, however note that when summing them up again from N_0 to ∞ telescopic cancellations occur turning the whole sum into

$$\frac{1}{2}(\ln\det(Id_{d_s} - \Psi(x_{N_0})) - \ln\det(Id_{d_s} - \Psi(h(x_{N_0})))) -$$
$$- \lim_{n\to\infty} \frac{1}{2}(\ln\det(Id_{d_s} - \Psi(h(x_n))) - \ln\det(Id_{d_s} - \Psi(x_n))) \quad (4.3)$$

In the following we show that there exists a $\theta \in (0,1)$, such that for all large enough n

$$|\ln\det(Id_{d_s} - \Psi(x_n)) - \ln\det(Id_{d_s} - \Psi(h(x_n)))| \leq C \cdot \theta^{-n}$$

hence the limit above exists, is zero and the whole difference can be estimated as $C \cdot \theta^{-N_0}$.
First, to conclude that the regularity of this difference is the same as the regularity of the elements of $\Psi(x)$, we will show that the determinants are bounded away from 0.

Lemma 4.4. *There is a uniform constant $C > 0$, such that* $\det(Id_{d_s} - \Psi(x)) > C$ *for all x.*

Proof. Recall that $\Psi(x)_{i,j} = \sum_{k=1}^{d_u} \left\langle \underline{f}_i(x), \underline{e}_k(x) \right\rangle \left\langle \underline{f}_j(x), \underline{e}_k(x) \right\rangle$, i.e. $\Psi(x)$ is a Gram matrix, therefore it has only nonnegative eigenvalues. Also note that $\Psi(x)$ is a symmetric matrix, hence its largest eigenvalue is $\max_{\underline{v}:\|\underline{v}\|=1} \left|\underline{v}^T \Psi(x) \underline{v}\right|$. By the definition of $\Psi(x)$ this last expression is nothing but $\left\|O_u^T(x) O_s(x) \underline{v}\right\|$. Let $\underline{v}$ be any vector of unit length in E_x^s with coordinates $\{v_k\}_{k=1}^{d_s}$ in the basis $\{\underline{f}_k(x)\}_{k=1}^{d_s}$. Then $(O_u^T(x) O_s(x) \underline{v})_i = \sum_{k=1}^{d_s} \left\langle \underline{f}_k(x), \underline{e}_i(x) \right\rangle v_k$. So the length of the vector $O_u^T(x) O_s(x) \underline{v}$ is equal to the length of the vector $\sum_{i=1}^{d_u} \sum_{k=1}^{d_s} v_k \left\langle \underline{f}_k(x), \underline{e}_i(x) \right\rangle \underline{e}_i(x)$.

But this last one is the orthogonal projection of the unit vector $\sum_{k=1}^{d_s} v_k \underline{f}_k(x)$ onto $T_x W$, which – by uniform transversality of stable and unstable cones – has length at most $C_0 < 1$. This means that all eigenvalues of $\Psi(x)$ are uniformly less than 1, which proves our statement. $\square$

Note that the quantity $C(x)$ in Lemma 4.3 involves all the geometric relations between the considered tangent spaces and it does not depend on the actual choice of any of the previous bases. Therefore we are free to choose these bases according to our needs.
In view of how the elements of $\Psi(x)$ look like, it is enough to discuss the regularity of the scalar products $\left\langle \underline{e}_i(x), \underline{f}_j(x) \right\rangle$. Identify the tangent spaces in x_n and $h(x_n)$ by parallel translation and then consider the difference

$$\begin{aligned} &|\langle \underline{e}_i(x_n), \underline{f}_j(x_n)\rangle - \langle \underline{e}_i(h(x_n)), \underline{f}_j(h(x_n))\rangle| \leq \\ &\quad \leq |\langle \underline{e}_i(x_n), \underline{f}_j(x_n) - \underline{f}_j(h(x_n))\rangle| + |\langle \underline{e}_i(x_n) - \underline{e}_i(h(x_n)), \underline{f}_j(h(x_n))\rangle| \end{aligned} \tag{4.4}$$

Since the curvature of stable manifolds is bounded by a global constant ((2.2) holds for the inverse dynamics too) the first term can be estimated as

$$\left\| \underline{f}_j(x_n) - \underline{f}_j(h(x_n)) \right\| \leq C \cdot d(x_n, h(x_n)) \leq C' \cdot \Lambda^{-n}.$$

Now to estimate the second term in (4.4) use that it is less than or equal to $\|\underline{e}_i(x_n) - \underline{e}_i(h(x_n))\|$. Choose an orthonormal basis $\{\underline{e}_i^n\}_{i=1}^{d_u}$ in $T_{x_n}W_1^n$. Iterating this k steps backwards (k is to be chosen later) by the tangent map $D_{x_n}F^{-k}$ gives the linearly independent vectors $\{\underline{e}_i^{n-k}\}_{i=1}^{d_u}$ in $T_{x_{n-k}}W_1^{n-k}$. At the point $h(x_{n-k})$ identify the origins of $T_{h(x_{n-k})}\mathcal{M}$ and $T_{x_{n-k}}\mathcal{M}$ to make sense of the vectors $\{\underline{e}_i^{n-k}\}_{i=1}^{d_u}$ in the first tangent space. By uniform transversality of stable and unstable cones we have that $T_{h(x_{n-k})}M = T_{h(x_{n-k})}W_2^{n-k} \oplus E^s_{h(x_{n-k})}$, hence each $\underline{e}_i^{n-k}$ can be uniquely decomposed as $\underline{v}_i + \underline{s}_i$ with vectors from the previous subspaces. Note that if $i \neq j$ then $\underline{v}_i \neq \underline{v}_j$ either, because otherwise $\underline{e}_i^{n-k} - \underline{e}_j^{n-k}$ would lie in $E^s_{h(x_{n-k})}$ and in $T_{x_{n-k}}W_1^{n-k}$ at the same time. This is of course impossible since these subspaces are transversal.
Now consider the vectors $D_{h(x_{n-k})}F^k\underline{v}_i$. In the rest of the proof we will show that these vectors are close to form an orthonormal basis in $T_{h(x_n)}W_2^n$ and also that after the necessary corrections the resulting basis vectors are close to the corresponding basis vectors in $T_{x_n}W_1^n$. We consider the deviation

$$\begin{aligned} &\left\|D_{h(x_{n-k})}F^k\underline{v}_i - \underline{e}_i^n\right\| \leq \\ &\leq \left\|D_{h(x_{n-k})}F^k\underline{v}_i - D_{h(x_{n-k})}F^k\underline{e}_i^{n-k}\right\| + \left\|D_{h(x_{n-k})}F^k\underline{e}_i^{n-k} - \underline{e}_i^n\right\| = \\ &= \left\|D_{h(x_{n-k})}F^k\underline{s}_i\right\| + \left\|D_{h(x_{n-k})}F^k\underline{e}_i^{n-k} - D_{x_{n-k}}F^k\underline{e}_i^{n-k}\right\| \end{aligned} \tag{4.5}$$

First of all the length of $\underline{e}_i^n$ is 1 by definition, so due to uniform hyperbolicity $\left\|\underline{e}_i^{n-k}\right\| \leq \Lambda^{-k}$. Then by the uniform transversality of stable and unstable cones we know that there is a constant C_0 such that $\|\underline{s}_i\| \leq C_0 \cdot \Lambda^{-k}$. Since $\underline{s}_i$ is a vector in $E^s_{h(x_{n-k})}$ it gets contracted exponentially under the action of the tangent map, so the first term in (4.5) is less than $C_0 \cdot \Lambda^{-2k}$.
To estimate the second term we will use the C^2 regularity of the dynamics. Due to this there are positive constants C_1 and C_2 such that for all x and y, which are in the same smoothness component of F

$$\|D_xF\| \leq C_1 \quad \text{and} \quad \|D_xF\underline{e} - D_yF\underline{e}\| \leq C_2 \cdot d(x,y) \tag{4.6}$$

where $\underline{e}$ is an arbitrary unit vector. Using the additional information that in our case $\underline{e}_i^{n-k}$ is an unstable vector of length at most Λ^{-k} and the points x_{n-k} and $h(x_{n-k})$ are on the same stable manifold by definition of h with distance at most $C \cdot \Lambda^{-(n-k)}$, we have the estimate

$$\left\| D_{h(x_{n-k})} F^k \underline{e}_i^{n-k} - D_{x_{n-k}} F^k \underline{e}_i^{n-k} \right\| \leq \frac{C_1^k - 1}{C_1 - 1} C_2 \cdot C \cdot \Lambda^{-n} \tag{4.7}$$

Hence we can estimate the deviation in (4.5) as

$$\begin{aligned} \left\| D_{h(x_{n-k})} F^k \underline{v}_i - \underline{e}_i^n \right\| \leq C_0 \cdot \Lambda^{-2k} + \frac{C_1^k - 1}{C_1 - 1} C_2 \cdot C \cdot \Lambda^{-n} < \\ < C_0 \cdot \Lambda^{-2k} + B \cdot \Lambda^{-n} \cdot C_1^k \end{aligned} \tag{4.8}$$

where $B = \frac{C_2 \cdot C}{C_1 - 1}$. Now we choose k for this bound to be optimal, which turns out to be $k = \frac{\ln\left(\frac{C_0 \ln \Lambda^2 \cdot \Lambda^n}{B \ln C_1}\right)}{\ln(C_1 \Lambda^2)}$. For this optimal choice of k we get the estimate

$$\left\| D_{h(x_{n-k})} F^k \underline{v}_i - \underline{e}_i^n \right\| \leq Const(C, C_1, C_2, \Lambda) \cdot \Lambda^{-\frac{2}{2 + \frac{\ln C_1}{\ln \Lambda}} n} = Const(C, C_1, C_2, \Lambda) \cdot \theta^n \tag{4.9}$$

where

$$\theta := \Lambda^{-\frac{2}{2 + \frac{\ln C_1}{\ln \Lambda}}} \in (0, 1) \tag{4.10}$$

In this sense our candidates $\underline{v}_i^n := D_{h(x_{n-k})} F^k \underline{v}_i$ for forming an orthonormal basis in $T_{h(x_n)} W_2^n$ are close to the corresponding vectors of the original basis in $T_{x_n} W_1^n$, yet they are not necessarily orthonormal. What remains is to show that the corrections needed to make them orthonormal are small, i.e. less than $\tilde{C} \cdot \theta^n$ for some constant $\tilde{C} > 0$.
From (4.9) it follows that all $\underline{v}_i^n$'s have almost unit length

$$\left| \, \|\underline{v}_i^n\| - 1 \right| \leq \|\underline{v}_i^n - \underline{e}_i^n\| \leq C \cdot \theta^n \tag{4.11}$$

A combination of (4.9) and (4.11) shows that for all $i \neq j$ the vectors $\underline{v}_i^n$ and $\underline{v}_j^n$ (after normalization) are almost perpendicular

$$\begin{aligned} \frac{1}{\|\underline{v}_i^n\| \left\|\underline{v}_j^n\right\|} \left| \left\langle \underline{v}_i^n, \underline{v}_j^n \right\rangle \right| \leq \frac{\|\underline{v}_i^n - \underline{e}_i^n\|}{\|\underline{v}_i^n\|} + \frac{\left\|\underline{v}_j^n - \underline{e}_j^n\right\|}{\|\underline{v}_i^n\| \left\|\underline{v}_j^n\right\|} \leq \\ \leq \frac{C \cdot \theta^n}{1 - C \cdot \theta^n} + \frac{C \cdot \theta^n}{(1 - C \cdot \theta^n)^2} \leq \tilde{C} \cdot \theta^n \end{aligned} \tag{4.12}$$

provided that n is large enough.
We proceed with Gram-Schmidt orthogonalization to construct an orthonormal basis in $T_{h(x_n)} W_2^n$ from the $\underline{v}_i^n$'s. To guarantee that the resulting vectors differ from the corresponding basis vectors in $T_{x_n} W_1^n$ by vectors with length at most $const \cdot \theta^n$ it is enough to check that at each step of the orthogonalization the correction needed has length at most $const \cdot \theta^n$. First define $\underline{w}_i^1$'s to be the normalized versions of the vectors $\underline{v}_i^n$ respectively (here the superscript of $\underline{w}_i$ refers to the first step of the orthogonalization process). The normalizing corrections have lengths at most $C \cdot \theta^n$ due to (4.11). Fix $\underline{w}_1^1$ to be the first vector of the orthonormal basis being constructed, then define $\underline{w}_i^2 = \underline{w}_i^1 - \left\langle \underline{w}_i^1, \underline{w}_1^1 \right\rangle \underline{w}_1^1$ for $i = 2, \ldots, d_u$.

These are perpendicular to $\underline{w}_1^1$ and the corrections needed to ensure this have lengths at most $\tilde{C}\cdot\theta^n$ due to (4.12). Then we normalize these vectors. Since $\left\|\underline{w}_i^1 - \langle \underline{w}_i^1, \underline{w}_1^1\rangle \underline{w}_1^1\right\|^2 = \left\|\underline{w}_i^1\right\|^2 - \left\|\langle \underline{w}_i^1, \underline{w}_1^1\rangle \underline{w}_1^1\right\|^2 = 1 - \langle \underline{w}_i^1, \underline{w}_1^1\rangle^2 \geq 1 - \tilde{C}^2\cdot\theta^{2n}$ again due to (4.12), the normalizing corrections have lengths at most $\tilde{C}\cdot\theta^n$. We may iterate this process with the remaining vectors to get the desired basis. The only thing we have to check for this, is that the normalized vectors from the last step are still pairwise almost perpendicular.

$$|\langle \underline{w}_i^1 - \langle \underline{w}_i^1, \underline{w}_1^1\rangle \underline{w}_1^1, \underline{w}_j^1 - \langle \underline{w}_j^1, \underline{w}_1^1\rangle \underline{w}_1^1\rangle| = |\langle \underline{w}_i^1, \underline{w}_j^1\rangle - \langle \underline{w}_i^1, \underline{w}_1^1\rangle \langle \underline{w}_j^1, \underline{w}_1^1\rangle| \leq \\ \leq \tilde{C}\cdot\theta^n + \tilde{C}^2\cdot\theta^{2n} \quad (4.13)$$

This holds for the vectors before the normalization. After normalizing them, their scalar products can be estimated by using the lower bound on their lengths established previously and the estimate will be $\frac{\tilde{C}\cdot\theta^n + \tilde{C}^2\cdot\theta^{2n}}{1-\tilde{C}^2\cdot\theta^{2n}} = \frac{\tilde{C}\cdot\theta^n}{1-\tilde{C}\cdot\theta^n}$. Again if n is sufficiently large then this is clearly at most $const\cdot\theta^n$ so the iteration can be done resulting in an orthonormal basis what we wanted. □

Now we establish bounds on the Jacobian of the holonomy map in terms of the initial geometry. More precisely we show that $\mathcal{J}_{W_1}h(x)$ is close to 1 if the distance $d(x,h(x))$ is small and the tangent spaces of the u-manifolds W_1 and W_2 are close enough in a certain sense.

Lemma 4.5. *Given the u-manifolds W_1 and W_2 assume that their tangent spaces are close to each other in the following sense. If $\underline{e}$ is a unit vector in T_xW_1 and its parallel transport to $h(x)$ has the decomposition $\underline{e} = \underline{v} + \underline{s} \in T_{h(x)}W_2 \oplus E^s_{h(x)}$ then $\|\underline{s}\| \leq \delta$ for some $\delta > 0$ small number. We claim that there are constants $\tilde{C}, \hat{C} > 0$ and $a \in (0,1)$ such that*

$$|\ln(\mathcal{J}_{W_1}h)(x)| \leq \tilde{C}\cdot d(x,h(x))^a + \hat{C}\cdot\delta.$$

Proof. Recall that we had the infinite sum representation of the Jacobian

$$|\ln(\mathcal{J}_{W_1}h)(x)| = \left|\sum_{i=0}^{\infty} \ln(\mathcal{J}_{W_1^i}F)(x_i) - \ln(\mathcal{J}_{W_2^i}F)(h(x_i))\right|,$$

and that each difference can be rewritten in terms of the total Jacobian, the Jacobian along stable manifolds and another term (which we denoted by $C(x)$) containing the geometric relations (cf. Lemma 4.3).

!!!Inner Comment!!! 4.6. *A következő mondat csúsztat egy kicsit. Az állítás ugyan igaz, de ahová hivatkozunk ott nincs igazán leírva.*

We have already showed in the proof of Lemma 4.2 that the whole sum of the differences of the total Jacobian and the Jacobian restricted to the stable manifold is bounded by $C\cdot d(x,h(x))$ for some constant $C > 0$. Hence it is enough to give the bound on the differences $\ln C(x) - \ln C(h(x))$. Again by our results from Lemma 4.2 it is enough to estimate the sum

$$\sum_{n=0}^{\infty} \|\underline{e}_i(x_n) - \underline{e}_i(h(x_n))\|,$$

where $\{\underline{e}_i(x_n)\}_{i=1}^{d_u}$ is a parallel translation of an orthonormal basis in $T_{x_n}W_1^n$ to $h(x_n)$ and $\{\underline{e}_i(h(x_n))\}_{i=1}^{d_u}$ is an orthonormal basis in $T_{h(x_n)}W_2^n$. As previously, we are free to choose these bases according to our needs. Take a unit vector $\underline{e} \in T_xW_1$ and for a fixed $n > 0$ consider its image $D_xF^n\underline{e}$. Its parallel translate to $h(x_n)$ can be decomposed as $\underline{v}^n + \underline{s}^n \in T_{h(x_n)}W_2^n \oplus E^s_{h(x_n)}$. If we do this the other way around, i.e. we first take the parallel translate of $\underline{e}$ to $h(x)$ and then consider the image $D_{h(x)}F^n\underline{e}$, we get the decomposition $D_{h(x)}F^n\underline{e} = \underline{w}^n + \underline{p}^n \in T_{h(x_n)}W_2^n \oplus E^s_{h(x_n)}$ (here $\underline{w}^n = D_{h(x)}F^n\underline{v}$ actually). Now we estimate the difference

$$\begin{aligned} &\|D_xF^n\underline{e} - \underline{v}^n\| \le \\ &\le \left\|D_xF^n\underline{e} - D_{h(x)}F^n\underline{e}\right\| + \left\|D_{h(x)}F^n\underline{e} - D_{h(x)}F^n\underline{v}\right\| + \left\|D_{h(x)}F^n\underline{v} - \underline{v}^n\right\|, \end{aligned} \tag{4.14}$$

in the order of these three terms as follows.

1. For the first term use (4.6) (the C^2 regularity of the dynamics) and the uniform contraction in the stable direction to conclude that

$$\left\|D_xF^n\underline{e} - D_{h(x)}F^n\underline{e}\right\| \le C_1^{n-1} \cdot C_2 d(x, h(x)) \frac{1 - 1/\Lambda^n}{1 - 1/\Lambda}.$$

2. For the second term use our assumption on the closeness of the tangent spaces of our u-manifolds together with the uniform contraction in the stable direction to deduce that

$$\left\|D_{h(x)}F^n\underline{e} - D_{h(x)}F^n\underline{v}\right\| = \left\|D_{h(x)}F^n\underline{s}\right\| \le \delta \cdot \Lambda^{-n}.$$

3. For the final term we introduce the operator $P : T_{h(x_n)}\mathcal{M} \to T_{h(x_n)}W_2^n$, which is the projection along $E^s_{h(x_n)}$ onto $T_{h(x_n)}W_2^n$. By uniform transversality of stable and unstable cones it follows that $\|P\| \le C_t$ for some constant $C_t > 0$ determined by the minimal angle between the conefields. The estimate then goes similarly as for the first term.

$$\left\|D_{h(x)}F^n\underline{v} - \underline{v}^n\right\| = \left\|PD_{h(x)}F^n\underline{e} - PD_xF^n\underline{e}\right\| \le C_tC_1^{n-1} \cdot C_2 d(x, h(x)) \frac{1 - 1/\Lambda^n}{1 - 1/\Lambda}.$$

The vector $\underline{e}$ has length 1 so due to the uniform expansion $\|D_xF^n\underline{e}\| \ge \Lambda^n$ and therefore we get that

$$\left\|\frac{D_xF^n\underline{e}}{\|D_xF^n\underline{e}\|} - \frac{\underline{v}^n}{\|D_xF^n\underline{e}\|}\right\| \le (C_t + 1)C_2 \frac{1}{\Lambda - 1}\left(\frac{C_1}{\Lambda}\right)^{n-1} d(x, h(x)) + \delta \cdot \Lambda^{-2n}. \tag{4.15}$$

Notice that $\underline{e} \in T_xW_1$ was an arbitrary unit vector, so every unit vector from $T_{x_n}W_1^n$ differs from its projection along $E^s_{h(x_n)}$ onto $T_{h(x_n)}W_2^n$ by at most the amount in (4.15). Hence, similar to our Gram-Schmidt argument at the end of the proof of Lemma 4.2, we can find two orthonormal bases in $T_{x_n}W_1^n$ and $T_{h(x_n)}W_2^n$ respectively, such that the difference of the corresponding basis vectors is at most $C_4[\left(\frac{C_1}{\Lambda}\right)^{n-1} d(x, h(x)) + \delta \cdot \Lambda^{-2n}]$, where $C_4 > 0$ is a global constant. Clearly this bound gets worse as we consider higher and higher iterates, but recall that we have another estimate on the tail from the proof of Lemma 4.2. This later one can be extrapolated for any $n \geq 0$ and so

$$\|\underline{e}_i(x_n) - \underline{e}_i(h(x_n))\| \leq \min\{C_4[\left(\frac{C_1}{\Lambda}\right)^{n-1} d(x,h(x)) + \delta \cdot \Lambda^{-2n}], C_3 \cdot \theta^n\}.$$

To estimate this we solve the equation $C_4 d(x,h(x)) \left(\frac{C_1}{\Lambda}\right)^{n-1} = C_3 \cdot \theta^n$ in n. We denote the solution by n_0 and also introduce the quantity $a_0 = \log_{C_1/\Lambda}(1/\theta) > 0$. A simple calculation shows that with these notations

$$\left(\frac{C_1}{\Lambda}\right)^{n_0} = \left(\frac{C_1 C_3}{C_4 \Lambda d(x,h(x))}\right)^{1/(1+a_0)}.$$

Hence for any index i we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \|\underline{e}_i(x_n) - \underline{e}_i(h(x_n))\| \leq \\
&\leq C_4\delta + \sum_{n=1}^{n_0} C_4[\left(\frac{C_1}{\Lambda}\right)^{n-1} d(x,h(x)) + \delta \cdot \Lambda^{-2n}] + \sum_{n_0+1}^{\infty} C_3\theta^n = \\
&= C_4 d(x,h(x))\frac{(C_1/\Lambda)^{n_0} - 1}{C_1/\Lambda - 1} + C_4\delta\frac{1 - 1/\Lambda^{2(n_0+1)}}{1 - 1/\Lambda^2} + C_3\theta^{n_0+1}\frac{1}{1-\theta} \leq \qquad (4.16)\\
&\leq \frac{\Lambda}{C_1 - \Lambda}\left(\frac{C_1 C_3}{\Lambda}\right)^{1/(1+a_0)} (C_4 d(x,h(x)))^{a_0/(1+a_0)} + C_4\delta\frac{\Lambda^2}{\Lambda^2 - 1} + \\
&+ \frac{\Lambda}{C_1\theta(1-\theta)}(C_4 d(x,h(x)))^{a_0/(1+a_0)} = \\
&= \tilde{C}' \cdot d(x,h(x))^a + \hat{C} \cdot \delta,
\end{aligned}$$

where $a = a_0/(1+a_0) \in (0,1)$. This completes the proof. □

We also want to understand how the value of the Jacobian of the holonomy map varies along its domain. We note that $\ln(\mathcal{J}_{W_1} h)(x)$ is *not* Hölder continuous in general. This is because two closeby points may be separated by a singularity of the dynamics so their futures are very different and hence their local stable manifolds can also be very different. Due to this phenomenon the regularity should be characterized with respect to the symbolic distance of the points.

Proposition 4.7. *The logarithm of the Jacobian of the holonomy map is dynamically Hölder continuous, i.e. for every $x, y \in W_1'$,*

$$|\ln(\mathcal{J}_{W_1} h)(x) - \ln(\mathcal{J}_{W_1} h)(y)| \leq C_0 \cdot \theta_h^{s_+(x,y)}$$

for some constants $C_0 > 0$ and $\theta_h \in (0,1)$.

Proof. To understand the regularity of the difference $|\ln(\mathcal{J}_{W_1} h)(x) - \ln(\mathcal{J}_{W_1} h)(y)|$, again by Proposition 4.1, one has to deal with the following infinite sum.

$$\left|\sum_{i=0}^{\infty} \ln(\mathcal{J}_{W_1^i} F)(x_i) - \ln(\mathcal{J}_{W_1^i} F)(y_i) + \ln(\mathcal{J}_{W_2^i} F)(h(y_i)) - \ln(\mathcal{J}_{W_2^i} F)(h(x_i))\right| \qquad (4.17)$$

We will use triangular inequality to estimate the terms but we gather them differently for small and large indices. For small values of i we consider the differences along u-manifolds and use the distorsion bound (2.3) to estimate them. For large values of i we consider differences along stable manifolds and use the tail bound from Lemma 4.2 to give bounds on them.
As we pointed out before if x and y are on the same u-manifold U, then $d_U(x,y) \le C \cdot \Lambda^{-s_+(x,y)}$. Choose $N_0 = \frac{s_+(x,y)}{2}$. Then there is a connected component of $F^{N_0}U$ that contains both x_{N_0} and y_{N_0}, which is itself a u-manifold. Since $s_+(F^{N_0}x, F^{N_0}y) = \frac{s_+(x,y)}{2}$, we have that $d_{F^{N_0}U}(F^{N_0}x, F^{N_0}y) \le C \cdot \Lambda^{-\frac{s_+(x,y)}{2}}$. The points x and $h(x)$ (and also y and $h(y)$) are on the same stable manifold by definition of the holonomy map and so $s_+(x,y) = s_+(h(x), h(y))$. Therefore the same idea applies to the points $h(x)$ and $h(y)$, hence using the distortion bound (2.3) allows us to estimate the first N_0 terms in (4.17) to be at most $2C' \cdot \Lambda^{-\frac{s_+(x,y)}{2}}$ (observe that the number of terms depends on the points x and y). The rest of the sum is at most $2C\theta^{N_0}$ by Lemma 4.2, with $\theta \in (0,1)$ defined by (4.10). Therefore

$$|\ln(\mathcal{J}_{W_1}h)(x) - \ln(\mathcal{J}_{W_1}h)(y)| \le 2C'\Lambda^{-\frac{s_+(x,y)}{2}} + 2C\theta^{\frac{s_+(x,y)}{2}} \le C_0\theta_h^{s_+(x,y)}, \quad (4.18)$$

where $\theta_h = \sqrt{\theta} \in (0,1)$. □

5 Coupling

The heuristic argument of proving exponential decay of correlations by coupling is quite picturesque. Starting from two standard pairs (W_1, ν_1) and (W_2, ν_2) their forward iterates will be expanded by the dynamics and cut by the singularities. After some steps due to our assumption (A5) certain components of $F^n(W_1)$ and $F^n(W_2)$ will be so close to each other, that several stable manifolds intersect both of them. The points $x_1 \in F^n(W_1)$ and $x_2 \in F^n(W_2)$ that lie on the same stable manifold will stay close to each other in the future, moreover their distance will converge to zero exponentially fast. This motivates coupling the measures (or at least a fraction of them) they carry. To make sure that the same amount of measure is coupled on both sides we introduce one extra (artificial) dimension, which will be responsible to record the amount of measure being coupled.

Definition 5.1. *For each u-manifold W in a standard family we define $\hat{W} := W \times [0,1]$ and equip it with the probability measure $\hat{\nu}$ defined by $d\hat{\nu}(x,t) = d\nu(x)dt = \rho(x)dxdt$. We call then $(\hat{W}, \hat{\nu})$ the* ***cylindrical extension*** *of the standard pair (W, ν). The extension of the map F and any observable f from W to $\hat{W}$ is done in the obvious way: $F(x,t) = (F(x), t)$ and $f(x,t) = f(x)$.*

The rest of the paper is dedicated to prove the following key lemma.

Lemma 5.2 (Coupling lemma)**.** *Let $\mathcal{G} = \{(W_i, \nu_i)\}_{i\in I}$ and $\mathcal{E} = \{(W_j, \nu_j)\}_{j\in J}$ be two proper standard families with measures $\mu_{\mathcal{G}}$ and $\mu_{\mathcal{E}}$ induced by them. Then there exists a bijection (a coupling map) $\Theta : \cup_{i\in I}\hat{W}_i \to \cup_{j\in J}\hat{W}_j$ that preserves measure, i.e. $\Theta(\hat{\mu}_{\mathcal{G}}) = \hat{\mu}_{\mathcal{E}}$, and a coupling time function $\Gamma : \cup_{i\in I}\hat{W}_i \to \mathbb{N}$ such that*

- *if* $(x,t) \in \cup_{i\in I} \hat{W}_i$*,* $\Theta(x,t) = (y,s) \in \hat{W}_j$ *for some* j *and* $m = \Gamma(x,t)$*, then the points* $F^m(x)$ *and* $F^m(y)$ *lie on the same stable manifold,*
- *there is a uniform exponential tail bound on the function* Γ*, i.e. we have* $\hat{\mu}_{\mathcal{G}}((x,t) : \Gamma(x,t) > n) \leq C_\Gamma \theta_\Gamma^n$ *for some constants* $C_\Gamma > 0$ *and* $\theta_\Gamma \in (0,1)$*.*

This is the key lemma that implies all fine statistical properties for the class of systems satisfying (A1)-(A5). For a list of statements and their proofs we suggest the reader to look at [8, Section 4]. Here we include the proof of the equidistribution property and the exponential

Proof of Theorem 2.6. As we already stated in Remark 3.19 the measurable partition of $\mathcal{M}$ into local unstable manifolds, with probability measures on them induced by the invariant measure μ, is a proper standard family. We denote this special, F-invariant family by $\mathcal{E}$ and so $\mu_{\mathcal{E}} = \mu$. We apply the coupling lemma to the proper families $\mathcal{G}$ and $\mathcal{E}$ to get a coupling map Θ between the two and a corresponding coupling time Γ. Then we take a dynamically Hölder continuous function f and consider the difference

$$\begin{aligned}
&\int_{\mathcal{M}} f \circ F^n \, d\mu_{\mathcal{G}} - \int_{\mathcal{M}} f \circ F^n \, d\mu_{\mathcal{E}} = \\
&= \int_{\hat{\mathcal{G}}} f(F^n(x,t)) \, d\hat{\mu}_{\mathcal{G}} - \int_{\hat{\mathcal{E}}} f(F^n(y,s)) \, d\hat{\mu}_{\mathcal{E}} = \\
&= \int_{\hat{\mathcal{G}}} [f(F^n(x,t)) - f(F^n(\Theta(x,t)))] \, d\hat{\mu}_{\mathcal{G}}
\end{aligned} \tag{5.1}$$

If $\Theta(x,t) = (y,s)$ and $m = \Gamma(x,t) \leq n$, then by the first part of the coupling lemma we know that $F^n(x)$ and $F^n(y)$ are on the same stable manifold and $s_-(F^n(x), F^n(y)) > n - m$. So using the dynamical Hölder continuity of f we have that

$$|f(F^n(x,t)) - f(F^n(\Theta(x,t)))| \leq K_f \theta_f^{n-m}.$$

Now decompose the final integral in (5.1) by the partition $\hat{\mathcal{G}} = \{(x,t) | \Gamma(x,t) \leq n/2\} \cup \{(x,t) | \Gamma(x,t) > n/2\}$. Then (5) implies

$$\int_{\{(x,t)|\Gamma(x,t)\leq n/2\}} [f(F^n(x,t)) - f(F^n(\Theta(x,t)))] \, d\hat{\mu}_{\mathcal{G}} \leq K_f \theta_f^{n/2},$$

and for the other half we have the following bound by the second part of the coupling lemma

$$\int_{\{(x,t)|\Gamma(x,t)> n/2\}} [f(F^n(x,t)) - f(F^n(\Theta(x,t)))] \, d\hat{\mu}_{\mathcal{G}} \leq 2 \|f\|_\infty C_\Gamma \theta_\Gamma^{n/2}.$$

□

Proof of Theorem 2.7. We can write the function f as $f^+ - f^-$, (where f^+ is the positive and f^- is the negative part of f) and then decompose the correlation of f and g into two terms. It is important to note that the relation

$$|f^{+,-}(x) - f^{+,-}(y)| \leq |f(x) - f(y)|$$

is always true, hence both f^+ and f^- are dynamically Hölder continuous with the same parameters as f. This way, by paying a price of a factor of 2 at the end, we can assume that f is nonnegative. Using the invariance of the measure μ we can write the n-time correlation of f and g as

$$\int_{\mathcal{M}} (f \circ F^{-n/2})(g \circ F^{n/2})\, d\mu - \int_{\mathcal{M}} f\, d\mu \cdot \int_{\mathcal{M}} g\, d\mu. \tag{5.2}$$

As in the proof of the equidistribution property we will again use the special standard family $\mathcal{E} = \{W^u\}$ and $\mu_{\mathcal{E}} = \mu$. Denote by $\mathbb{E}_\mu(f \circ F^{-n/2}|\mathcal{E})$ the conditional expectation of the function $f \circ F^{-n/2}$ with respect to the partition $\mathcal{E}$ and the measure μ. Then we can write (5.2) as

$$\int_{\mathcal{M}} \left[(f \circ F^{-n/2}) - \mathbb{E}_\mu(f \circ F^{-n/2}|\mathcal{E})\right](g \circ F^{n/2})\, d\mu + \\ + \int_{\mathcal{M}} \mathbb{E}_\mu(f \circ F^{-n/2}|\mathcal{E})(g \circ F^{n/2})\, d\mu - \int_{\mathcal{M}} f\, d\mu \cdot \int_{\mathcal{M}} g\, d\mu. \tag{5.3}$$

To estimate the first integral observe that if x and y are two points of a local unstable manifold, then so are $F^{-n/2}(x)$ and $F^{-n/2}(y)$, moreover

$$s_+(F^{-n/2}(x), F^{-n/2}(y)) \geq n/2.$$

Since f is dynamically Hölder continuous, this implies that the oscillation of $f \circ F^{-n/2}$ along unstable manifolds is at most $K_f \theta_f^{n/2}$. Hence

$$\int_{\mathcal{M}} \left[(f \circ F^{-n/2}) - \mathbb{E}_\mu(f \circ F^{-n/2}|\mathcal{E})\right](g \circ F^{n/2})\, d\mu \leq K_f \|g\|_\infty \theta_f^{n/2}. \tag{5.4}$$

For the remaining terms we would like to apply Theorem 2.6. To this end we set a new standard family $\mathcal{G}$ to be $\mathcal{G} = \{W^u\}$, $\mu_{\mathcal{G}} \restriction_{W^u} = \mu_{\mathcal{E}} \restriction_{W^u}$ for every local unstable manifold W^u and

$$d\lambda_{\mathcal{G}}(i) = \frac{\mathbb{E}_\mu(f \circ F^{-n/2}|\mathcal{E})(W_i^u}{\int_{\mathcal{M}} f\, d\mu} d\lambda_\mu(i),$$

i.e. we only change the factor measure of the special standard family $\mathcal{E}$ but keep everything else the same. Note that we also normalized the new measure, so it is still a probability measure. Still we can not directly apply Theorem 2.6 because $\mathcal{G}$ may not be proper. We estimate $\mathcal{Z}_{\mathcal{G}}$ in the following. By definition of the new measure we have

$$\mu_{\mathcal{G}}(x|r(x) < \varepsilon) \leq \frac{\|f\|_\infty}{\int_{\mathcal{M}} f\, d\mu} \mu(x|r(x) < \varepsilon) \leq \frac{\|f\|_\infty}{\int_{\mathcal{M}} f\, d\mu} C_p \varepsilon,$$

using Remark 3.19. Hence

$$\mathcal{Z}_{\mathcal{G}} \leq \frac{\|f\|_\infty}{\int_{\mathcal{M}} f\, d\mu} C_p,$$

which is a uniform bound so we can use the growth lemma (Lemma 3.15) to compute the number of iterations needed for the family $\mathcal{G}$ to become proper. This number is

$$k = \log_\alpha C_p \frac{\|f\|_\infty}{\int\limits_{\mathcal{M}} f \, d\mu},$$

where $\alpha \in (0,1)$ is from the growth lemma. After this we can estimate the difference of the second and first terms from (5.3) as

$$\begin{aligned}
\int\limits_{\mathcal{M}} \mathbb{E}_\mu(f \circ F^{-n/2}|\mathcal{E})(g \circ F^{n/2}) \, d\mu - \int\limits_{\mathcal{M}} f \, d\mu \cdot \int\limits_{\mathcal{M}} g \, d\mu = \\
= \int\limits_{\mathcal{M}} f \, d\mu \Big(\int\limits_{\mathcal{M}} g \circ F^{n/2} \, d\mu_{\mathcal{G}} - \int\limits_{\mathcal{M}} g \, d\mu \Big) = \\
= \int\limits_{\mathcal{M}} f \, d\mu \Big(\int\limits_{\mathcal{M}} g \circ F^{n/2+k} \, d\mu_{F^k(\mathcal{G})} - \int\limits_{\mathcal{M}} g \, d\mu \Big) \leq \\
\leq \int\limits_{\mathcal{M}} f \, d\mu \cdot B_g \Theta_g^{n/2+k}, \quad (5.5)
\end{aligned}$$

because $F^k(\mathcal{G})$ is a *proper* standard family and so Theorem 2.6 can be applied. Note that $\Theta_g = (\max\{\theta_g, \theta_\Gamma\})^{1/2}$, in particular $\Theta_g > \theta_\Gamma$. The coupling of measures is heavily based on the growth of unstable manifolds characterized by Lemma 3.15. It will actually follow from the construction that $\theta_\Gamma \geq \alpha$. Therefore

$$\Theta_g^k = \Theta_g^{\log_\alpha C_p} \Theta_g^{\log_\alpha \|f\|_\infty - \log_\alpha \int f \, d\mu} \leq C_p \frac{\|f\|_\infty}{\int\limits_{\mathcal{M}} f \, d\mu},$$

hence (5.5) is at most $B_g C_p \|f\|_\infty \Theta_g^{n/2}$. This together with (5.4) implies that

$$\int\limits_{\mathcal{M}} f(g \circ F^n) \, d\mu - \int\limits_{\mathcal{M}} f \, d\mu \cdot \int\limits_{\mathcal{M}} g \, d\mu \leq K_f \|g\|_\infty \theta_f^{n/2} + B_g C_p \|f\|_\infty \Theta_g^{n/2}$$

leading to the proof of the theorem. □

!!!Inner Comment!!! 5.3. *A következő alfejezetben van a legnagyobb gond, alapvetően ezt kell helyre tenni. Itt egy kicsit több kommentet írok az állításokhoz. Az új érvelés vázlatosan valami ilyesmi:*

- *Minden δ_1-proper u-sokaságra építhető tégla. Ez kevesebb, mint Lemma 5.5, mert nem törődünk a mértékkel.*
- *Egyetlen egy tégláról megmutatjuk, hogy pozitív mértékű (R_σ), az ehhez tartozó stabil sokaságok uniója lesz a mágnes, σ. Ez lehet akár a korábban, a téglák konstrukciójánál használt érvelés adaptációja.*
- *Kompaktsági érveléssel megoldjuk, hogy véges sok tégla "lefedje" a δ_1-proper u-sokaságokat. Ha jól értem ez maradhat Lemma 5.6 annyi változtatással, hogy most sem beszélünk mértékekről.*

- *Lai-Sang azt mondja, hogy ha a véges sok tégla uniója F^R invariáns és irreducibilis, akkor igaz az átmetszési lemma. Az F^R invariancia nem is értem, hogy hogy nem állhatna fent, viszont az irreducibilitásról még nem tudtam meggyőzni magam teljesen...*
- *Feltéve, hogy idáig minden oké azt csináljuk, hogy elmondjuk az érvelést a szorzat rendszerre. Lai-Sang-ra való hivatkozással adódik, hogy*

  $$\mu \times \mu \restriction_{R_\sigma \times R_\sigma} (R > n) \leq C \cdot \theta^n.$$

  Ezért elég nagy, de fix n_0-ra a $\{0, \ldots, n_0\}$ időpontokban összesen csatolható mérték legalább $1 - C \cdot \theta^{n_0} =: p > 0$, így van egy olyan időpont is, amikor a csatolható mérték legalább $p/n_0 > 0$.
- *Az időpont bizonytalansága miatt a csatolási idő uniform becslésénél az eloszlást (néhai q_n) az n_0 hosszúságú blokkokra kell elmondani és nem az egyesével történő iteráltakra.*

Corollary 5.9-től az alfejezet maradhat változatlan.

5.1 Transitivity

We will use a special magnet on which the whole coupling procedure will take place. This special magnet is σ_W from Definition 3.10 with the following details. We consider an unstable manifold W that is a ball in its inner metric with radius $\max\{\frac{2}{C_t}(\frac{\delta_1}{3} + 4\delta_3), \delta_1\}$ (instead of radius δ_1). We build the special magnet on this W, and we set y_0 in Definition 3.10 to be the center of W (which is not necessarily a density point of the corresponding set W^1_∞ this time, but we do not need that here) and choose $r_W = \delta_1/3$ (the rest of the parameters in the definition remain the same). For this special magnet we will omit the subscript W. Then, just to recall it here

$$\sigma = \{W^s_{\delta_2}(x) | x \in W^1_\infty \cap B_W(y_0, \delta_1/3)\}.$$

This will be the set on which we will couple measures of standard pairs. Obviously to do so we have to guarantee that standard pairs cross σ (i.e. they intersect all the stable disks in σ), when they are iterated forward by the dynamics. In case the phase space is two dimensional, this was proven by Bunimovich, Chernov and Sinai in [2, Theorem 3.13], but for higher dimensional systems no proof was given so far. This extension is the purpose of the present subsection. First we introduce some notation. For any standard pair (W, ν) and $n \geq 0$ we denote by W^n_k the connected components of $F^n(W)$ that cross σ and $W^n_* := \cup_k F^{-n}(W^n_k \cap \sigma)$ the preimage of all the intersections.

Proposition 5.4 (Transitivity)**.** *There are constants $n_1 \geq 1$ and $d_1 > 0$ such that for any δ_1-proper standard pair (W, ν), and for any $n \geq n_1$ we have $\nu(W^n_*) \geq d_1$.*

The proof will consist of several steps, in particular a compactness argument, and uses the mixing property of the dynamics and its hyperbolicity.

Lemma 5.5. *Let (W, ν) be a δ_1-proper standard pair. Then there is a magnet σ_W and an adapted rectangle R_W, such that R_W has positive μ-measure, W crosses σ_W (i.e. W intersects all stable manifolds in σ_W) and $\nu(W \cap \sigma_W) > 0$.*

Proof. Due to our assumption there exists a point $x \in W$ such that $d_W(x, \partial W) > \delta_1$. Therefore we can use the construction given in Definition 3.10 to build a magnet σ_W and an adapted rectangle R_W on W. Then clearly W crosses σ_W and, because ν is uniformly equivalent to m_W, it also follows that $\nu(W \cap \sigma_W) > 0$. We are only left to argue that R_W has positive μ-measure. Since the jacobian of the holonomy map is uniformly bounded and arbitrarily small unstable manifolds form a full μ-measure set in $\mathcal{M}$, we can do a Lebesgue density argument. We use the advantage of the flexibility of the unstable size of R_W in Definition 3.10 to choose a sufficiently small r_W such that $\mu(R_W) > 0$. □

Note that we can also define a rectangle adapted to our special magnet σ. Simply consider the unstable manifold W, on which σ is built and define R_σ as in Definition 3.10 with base $B_W(x, \delta_1/3)$ except that here we do not require the W^u's to contain some ball. For R_σ it is enough for us that the W^u's are unstable manifolds overshadowing the base $B_W(x, \delta_1/3)$ with s-distance at most δ_3. Again by a Lebesgue density argument, using also that the jacobian of the holonomy map is uniformly bounded, one can show that $\mu(R_\sigma) > 0$ provided that δ_0 is small enough.
Now we turn to the compactness argument to gain advantage of the mixing property of the dynamics.

Lemma 5.6. *There exist a finite number of positive μ-measure rectangles $\{R_i\}_{i=1}^N$ (we denote by σ_i the collection of all stable manifolds forming intersections in R_i) such that, if (W, ν) is a δ_1-proper standard pair, then W crosses at least one of these σ_i's and the intersection has uniformly positive measure, i.e. $\exists j \in \{1, \dots, N\}$ and a global constant $q_0 > 0$ such that $\nu(W \cap \sigma_j) > q_0$.*

Proof. Consider the set

$A := \{W | W$ is an admissible u-manifold, moreover a δ_1-ball in its own metric$\}$.

This can be equipped with the Hausdorff-metric d_H and we take the closure of A with respect to this metric. By the compactness of $\mathcal{M}$ it follows that $\bar{A}$ is also compact. Since admissible u-manifolds have bounded curvature and stable and unstable cones are uniformly transversal it follows that there exists a small ($\ll \delta_3$) parameter δ_4 such that for any two u-manifolds $W, U \in A$ if $d_H(W, U) \leq \delta_4$ then U overshadows the central part $W_{\delta_1/3}$ of W (i.e. the ball in W with the same center as W but with radius $\delta_1/3$), moreover $d^s(W_{\delta_1/3}, U) \leq \delta_3$ (this statement is similar to the results of Lemma 3.7). We choose then a finite δ_4-net $\{W_i\}_{i=1}^N$ of $\bar{A}$ with all W_i's from A and construct rectangles $\{R_i\}_{i=1}^N$ on them as in Lemma 5.5.
Now if (W, ν) is δ_1-proper, i.e. $\exists W_{\delta_1} \subset W$, then obviously $W_{\delta_1} \in A$. Hence W will overshadow at least one of the W_i's with $d^s(W_{i,\delta_1/3}, W) \leq \delta_3$ and so intersects all the stable manifolds in σ_i. For all i we have $m_{W_i}(W_i \cap \sigma_i) > 0$ by Lemma 5.5, and since there are only finitely many indices $\exists q > 0$ such that $m_{W_i}(W_i \cap \sigma_i) > q$ for every i. Using the uniform equivalence of the measures m_W and ν, the fact that W is admissible (in particular its diameter is uniformly bounded from above) and that the jacobian of the holonomy map is also uniformly bounded, the statement of the lemma follows. □

So if (W, ν) is a sufficiently large standard pair then W crosses at least one of our rectangles, say R_j (more precisely it crosses σ_j, the collection of stable

manifolds from R_j). We will use the expression that R_j carries W.
The image of a rectangle $F^n(R_j)$ consists of a finite number of rectangles $\{R^n_{j,k}\}$, $k = 1, \dots, K(n)$ and the preimage of each is an s-subrectangle in R_j (i.e. for all $k \in \{1, \dots, K(n)\}$ and $x \in F^{-n}R^n_{j,k}$ we have $W^s(x) \cap R_j = W^s(x) \cap F^{-n}R^n_{j,k}$). Now if R_j carries W then each of the forward iterates $\{R^n_{j,k}\}$, $k = 1, \dots, K(n)$ carry a connected component W^n_k of $F^n(W)$ and different images carry different connected components. For our rectangles R_i, $i = 1, \dots, N$ there is a $q_1 > 0$ such that $\mu(R_i) \geq q_1$ for all i and let $\mu(R_\sigma) = q_2 > 0$. Because F is mixing it follows that there is an $n_0' > 0$ such that for every $n > n_0'$ we have $\mu(F^n(R_i) \cap R_\sigma) \geq \frac{q_1 q_2}{2}$ for every $i = 1, \dots N$. So for any j after a fixed number of iterations some of the rectangles $\{R^n_{j,k}\}$, $k = 1, \dots, K(n)$ will intersect our favourite rectangle R_σ and so the image of the u-manifold W carried by R_j will (likely to) intersect some of the stable manifolds of our special magnet σ. In the next lemma we show that if the number of iterations is large enough and the rectangle $R^n_{j,k}$ carrying W^n_k intersects R_σ then W^n_k not just intersects but actually crosses σ (i.e. W^n_k intersects all stable manifolds in σ).

Lemma 5.7. *Let (W, ν) be a δ_1-proper standard pair and R_j a rectangle from the finite collection that carries W. There exists an $n_0'' > 0$ such that for every $n \geq n_0''$ if a rectangle $R^n_{j,k}$ intersects R_σ, then W^n_k (the connected component of $F^n(W)$ carried by $R^n_{j,k}$) intersects all stable manifolds in σ.*

Proof. Assume that $x \in R^n_{j,k} \cap R_\sigma$. Then by the structure of the rectangles there is an unstable manifold $U_j(F^{-n}x)$ through the point $F^{-n}(x)$ - a "building block" of R_j - which was iterated forward by n steps. Recall that by the construction of R_j (explained in the proof of Lemma 5.5) $U_j(F^{-n}x)$ is a ball of radius $4r_j$ such that its central part with radius $2r_j$ overshadows a ball B_{j,r_j} with radius r_j on the u-manifold W_j on which R_j was built, and $d^s(B_{j,r_j}, U_j(F^{-n}x)) \leq \delta_3$. Similarly the u-manifold of the standard pair W overshadows a $\delta_1/3$-ball $B_{j,\delta_1/3} \subset W_j$, which contains B_{j,r_j} and $d^s(B_{j,\delta_1/3}, W) \leq \delta_3$. By Lemma 3.7 it follows that $B_{j,\delta_1/3}$ overshadows $U_j(F^{-n}x)$ (provided that r_j and δ_3 are small enough compared to δ_1) and $d^s(U_j(F^{-n}x), B_{j,\delta_1/3}) \leq 2(\frac{8}{C_t} + 1)\delta_3$. Then by the transitivity of the shadowing property we know that W also overshadows $U_j(F^{-n}x)$ with $d^s(U_j(F^{-n}x), W) \leq (\frac{16}{C_t} + 3)\delta_3$.
Now iterating $U_j(F^{-n}x)$ forward by the dynamics it gets expanded and possibly cut by singularities. Note that there is a ball $B_{2r_j}(F^{-n}x)$ (centered at $F^{-n}x$ with radius $2r_j$) such that $B_{2r_j}(F^{-n}x) \subset U_j(F^{-n}x)$. Choose $n_0 \geq \max_j\{\frac{1}{\ln \Lambda} \ln(\frac{\delta_1/3+\delta_3}{r_j})\}$, so that if $n \geq n_0$ the image $F^n(B_{2r_j}(F^{-n}x))$ will contain a ball around x with radius at least $2(\delta_1/3 + \delta_3)$ unless it is cut by a singularity at some intermediate step. But x also belongs to R_σ which implies that there is an unstable manifold $U_\sigma(x)$ through it with similar properties to what $U_j(F^{-n}x)$ had on R_j. Namely $U_\sigma(x)$ overshadows the central $\delta_1/3$-ball $B_{\sigma,\delta_1/3}$ of the unstable manifold, on which R_σ is built, and $d^s(B_{\sigma,\delta_1/3}, U_\sigma(x)) \leq \delta_3$. Hence there is a $\delta_1/3 + \delta_3$-ball in $U_\sigma(x)$ that contains the whole intersection $U_\sigma(x) \cap \sigma$ (which includes the point x too). Since local unstable manifolds are unique and they can not be cut by inverse singularities it follows that some part of the image $F^n(U_j(F^{-n}x))$ coincides with that $\delta_1/3 + \delta_3$-ball in $U_\sigma(x)$. Denote the preimage of this in $U_j(F^{-n}x)$ by $\tilde{U}$. We use the following result from the literature.

Lemma 5.8 ([6, Proposition 5.3]). *Let U be an admissible u-manifold, and W another u-manifold that overshadows U and $d^s(U,W) \leq \delta$. Let $\{U_n^1\}$ be a δ_2-filtration of U. Then $\forall n \geq 1$ and any connected component V of U_n^1 there is a connected domain $V' \subset W \setminus \mathcal{S}^{(n)}$ such that the u-manifold $F^n V'$ overshadows the admissable u-manifold $F^n V$, and $d^s(F^n V, F^n V') \leq \delta \Lambda^{-n}$.*

Apply this lemma with the choice $U := \tilde{U}$ and W the u-manifold of the standard pair. Due to our previous observations W overshadows $\tilde{U}$ with $d^s(\tilde{U}, W) \leq (\frac{16}{C_t} + 3)\delta_3$, this will be our δ in the lemma above. Observe that U is chosen in such a way that it remains connected within n iterations and its diameter does not increase over δ_0. Hence in its filtration $U_n^1 = U$ simply, which is one connected set, so $V = U_n^1$ in the lemma and the connected component W_k^n will play the role of $F^n V'$.
Therefore W_k^n overshadows $F^n\tilde{U}$ with $d^s(F^n\tilde{U}, W_k^n) \leq (\frac{16}{C_t} + 3)\delta_3 \cdot \Lambda^{-n}$. But recall that $F^n\tilde{U}$ is actually a $\delta_1/3 + \delta_3$-ball that overshadows the central $\delta_1/3$-ball $B_{\sigma,\delta_1/3}$ of the unstable manifold on which R_σ is built with s-distance at most δ_3. Hence again by the transitivity of the shadowing property W_k^n also overshadows $B_{\sigma,\delta_1/3}$ with s-distance at most $(\frac{16}{C_t} + 3)\delta_3 \cdot \Lambda^{-n} + \delta_3$. Clearly if n is large enough this is at most $2\delta_3$ which is less than δ_2 (using that $\delta_3 \leq \frac{1}{3}\delta_2$ by (3.4)), which implies that W_k^n indeed intersects all stable manifolds in σ. □

Remark: Note that the u-manifold W in Lemma 5.7 could be any unstable manifold along R_j, hence this Lemma implies that if $R_{j,k}^n$ is a (maximal) rectangle from the image $F^n(R_j)$ for some n large enough and it intersects R_σ, then all unstable manifolds in $R_{j,k}^n$ will actually cross the magnet σ.

Proof of Proposition 5.4. Recall that we have finitely many rectangles R_j (from Lemma 5.6) such that $\mu(R_j) \geq q_1$ for every j. Also we have our special rectangle R_σ with $\mu(R_\sigma) = q_2$. We choose n to be larger than $\max(n_0', n_0'')$ and we consider a rectangle R_j that carries the u-manifold W of our standard pair. We denote by $R_{j,k}^n$, $k = 1, \dots, K(n)$ those (maximal) rectangles in the image $F^n(R_j)$ that intersect R_σ and by W_k^n, $k = 1, \dots, K(n)$ the connected components of $F^n(W)$ carried by them. By our choice of n these rectangles and components are all crossing σ and we have that

$$\mu(F^n(R_j) \cap R_\sigma) = \sum_{k=1}^{K(n)} \mu(R_{j,k}^n \cap R_\sigma) \geq \frac{q_1 q_2}{2}$$

Since the measure μ is invariant, this holds also for the preimage, i.e.

$$\sum_{k=1}^{K(n)} \mu(F^{-n}(R_{j,k}^n \cap R_\sigma)) = \mu(R_j \cap F^{-n}(R_\sigma)) \geq \frac{q_1 q_2}{2} \tag{5.6}$$

Note that for each k the set $F^{-n}(R_{j,k}^n \cap R_\sigma)$ carries a subset of W, the F^n-image of which crosses the magnet σ. Denote by μ_U the conditional measure of μ when restricted on the u-manifold U and by m_U the normalized Lebesgue measure on U as usual.
Collect all unstable manifolds that were used in the construction of R_j and index

them by $\alpha \in A$ as W_α for some index set A. Then there is a factor measure λ on A such that we have

$$\mu(R_j) = \int\limits_A \mu_{W_\alpha}(W_\alpha \cap R_j)\, d\lambda(\alpha) \qquad \mu(R_j \cap F^{-n}(R_\sigma)) = \int\limits_A \mu_{W_\alpha}(W_\alpha \cap R_j \cap F^{-n}(R_\sigma))\, d\lambda(\alpha)$$

Together with (5.6) and the fact that μ is a probability measure this implies that

$$\frac{q_2}{2} \leq \frac{\mu(R_j \cap F^{-n}(R_\sigma))}{\mu(R_j)} = \frac{\int\limits_A \mu_{W_\alpha}(W_\alpha) \frac{\mu_{W_\alpha}(W_\alpha \cap R_j \cap F^{-n}(R_\sigma))}{\mu_{W_\alpha}(W_\alpha)}\, d\lambda(\alpha)}{\int\limits_A \mu_{W_\alpha}(W_\alpha) \frac{\mu_{W_\alpha}(W_\alpha \cap R_j)}{\mu_{W_\alpha}(W_\alpha)}\, d\lambda(\alpha)}$$

Since we assumed that μ is an SRB-measure there exists a constant $C_e > 1$ such that for each index α and for every measurable subset $B \subseteq W_\alpha$ we have

$$C_e^{-1} m_{W_\alpha}(B) \leq \frac{\mu_{W_\alpha}(B)}{\mu_{W_\alpha}(W_\alpha)} \leq C_e m_{W_\alpha}(B)$$

so the fractions appearing in the previous integrals are equivalent to the normalized Lebesgue measures of the sets in the numerators. Therefore $\frac{\mu_{W_\alpha}(W_\alpha \cap R_j)}{\mu_{W_\alpha}(W_\alpha)} \geq C_e^{-1} \cdot m_{W_\alpha}(W_\alpha \cap R_j) \geq C_e^{-1} q_0 > 0$, where $q_0 > 0$ is another constant coming from the facts that

- R_j was built in a way that there is a u-manifold W_j and an r_j-ball W_{j,r_j} inside it such that $m_{W_{j,r_j}}(W^1_{j,\infty}) > q_0' > 0$, where $W^1_{j,\infty}$ is the intersection of W_j with all the stable manifolds that were used in the construction of R_j,
- the diameter of W_j is at most δ_0,
- admissible u-manifolds have bounded curvature,
- the holonomy map is absolutely continuous

Using this $C_e^{-1} q_0$ lower bound in the previous denominator we get that

$$\frac{q_0 q_2}{2C_e} \leq \frac{\int\limits_A \mu_{W_\alpha}(W_\alpha) \frac{\mu_{W_\alpha}(W_\alpha \cap R_j \cap F^{-n}(R_\sigma))}{\mu_{W_\alpha}(W_\alpha)}\, d\lambda(\alpha)}{\int\limits_A \mu_{W_\alpha}(W_\alpha)\, d\lambda(\alpha)}$$

Here the right hand side is actually an expectation of some quantity with respect to the measure on A with density $\frac{\mu_{W_\alpha}(W_\alpha)\, d\lambda(\alpha)}{\int\limits_A \mu_{W_\alpha}(W_\alpha)\, d\lambda(\alpha)}$, hence we can conclude that there is an $\alpha_0 \in A$ such that

$$\frac{q_0 q_2}{2C_e} \leq \frac{\mu_{W_{\alpha_0}}(W_{\alpha_0} \cap R_j \cap F^{-n}(R_\sigma))}{\mu_{W_{\alpha_0}}(W_{\alpha_0})}$$

from which (again by the equivalence with the normalized Lebesgue measure) we get

$$m_{W_{\alpha_0}}(W_{\alpha_0} \cap R_j \cap F^{-n}(R_\sigma)) \geq \frac{q_0 q_2}{2C_e^2} \tag{5.7}$$

Now by following the exact same steps of the previous itemization, with the only modification of replacing the estimate in the first item by (5.7) it can be verified that

$$\nu(W_*^n) = \nu(\cup_{k=1}^{K(n)} F^{-n}(W_k^n \cap \sigma)) \geq C \cdot \frac{q_0 q_2}{2}$$

□

Corollary 5.9. *There are constants $n_0 \geq 1$ and $d_0 > 0$ such that for any proper standard family $\mathcal{G} = \{(W_i, \nu_i)\}_{i \in I}$ and any $n \geq n_0$ we have $\mu_{\mathcal{G}}(\cup_i W_{i,*}^n) \geq d_0$.*

Proof. If $\mathcal{G}$ is proper then by definition $\mathcal{Z}_{\mathcal{G}} \leq C_p$. Corollary 3.16 implies that for all $n \geq \chi \ln C_p$ we have $\mathcal{Z}_{\mathcal{G}_n} \leq c_3$. For a fixed such n we denote the image of the original standard family by $\mathcal{G}_n = \{W_j^n, \nu_j^n\}_{j \in J}$ for some index set J, where W_j^n's are the connected u-manifolds in the image, and ν_j^n's are the probability measures induced by the image of the original measures. There is also an induced probability factor measure on the index set J, which we denote by $d\lambda^n(j)$. With these notations $\mathcal{Z}_{\mathcal{G}_n} \leq c_3$ reads as

$$\forall \varepsilon > 0 \quad \int_J \nu_j^n(r_j^n(x) < \varepsilon)\, d\lambda^n(j) \leq c_3 \varepsilon.$$

We partition the set of indices as $J = J_1 \cup J_2$ in a way that $j \in J_1$ iff W_j^n is δ_1-proper. As $\mu_{\mathcal{G}_n}$ is a probability measure, considering the complementary event in the previous inequality and splitting the integral into two parts by the partition results in

$$\int_{J_1} \nu_j^n(r_j^n(x) > \varepsilon)\, d\lambda^n(j) + \int_{J_2} \nu_j^n(r_j^n(x) > \varepsilon)\, d\lambda^n(j) \geq 1 - c_3 \varepsilon. \tag{5.8}$$

Substituting $\varepsilon = \delta_1$ into (5.8) the second integral vanishes according to the definition of the partitions. Using that every ν_j^n is a probability measure and also the relation (3.6), we get the estimate

$$\int_{J_1} d\lambda^n(j) \geq \int_{J_1} \nu_j^n(r_j^n(x) > \varepsilon)\, d\lambda^n(j) \geq 1 - c_3 \varepsilon \geq 1/2.$$

The left hand side is actually the relative measure of the u-manifolds in $\mathcal{G}_n$ that are δ_1-proper. For each such manifold we can apply Proposition 5.4 to conclude that after another n_1 iterations at least a fixed $(d_1 > 0)$ amount of its measure will be on the magnet. Therefore the statement of the present corollary is true with $n_0 = \chi \ln C_p + n_1$ and $d_0 = d_1/2$. □

Remark 5.10. *Although the previous result is formulated only for proper standard families, if the standard family is not proper then Lemma 3.16 implies that after some number of iterations (this number is actually at most $\chi \ln \mathcal{Z}_{\mathcal{G}}$ by Corollary 3.16) it becomes proper, so Corollary 5.9 will hold by adding this number of iterations to n_0.*

5.2 Coupling of measures

The proof of Lemma 5.2 consists of two main steps. The first step relies on Corollary 5.9. We will describe how to couple the measures of two standard families once they are crossing the magnet. Then we will construct the coupling time function Γ and prove that it admits the desired exponential tail bound.
Consider two proper standard families $\mathcal{G} = \{(W_i, \nu_i)\}_{i\in I}$ and $\mathcal{E} = \{(W_j, \nu_j)\}_{j\in J}$. Slightly abusing the notation, for any $n \geq 0$ we will sometimes refer to the union $\cup_{i\in I} F^n(W_i)$ just by $F^n(\mathcal{G})$ simply. Imagine that after a certain number of iterations some of the components of $F^n(\mathcal{G})$ and $F^n(\mathcal{E})$ (we will index them by i and j respectively) are crossing the magnet σ, such that $\mu_{\mathcal{G}}(\cup_i W^n_{i,*}) \geq d_0$ and $\mu_{\mathcal{E}}(\cup_j W^n_{j,*}) \geq d_0$ (remember that this holds for every $n \geq n_0$ by Corollary 5.9).

!!!Inner Comment!!! 5.11. *Ezt ugye most nem állíthatjuk, csak azt, hogy minden n_0 lépésből álló blokkban megtörténik egyszer.*

In such a situation we couple a fixed amount of the measures in the following way. We consider the cylindrical extensions (as defined in Definition 5.1) of the components $W^n_{i,k}$ and $W^n_{j,l}$ that cross σ, to get the 'cylinders' $\hat{W}^n_{i,k} = W^n_{i,k} \times [0,1]$ and $\hat{W}^n_{j,l} = W^n_{j,l} \times [0,1]$. We would like to couple two sets of total measure $\frac{d_0}{2}$ in a way that the bijection between the points preserves measure. To this end we cut our cylinders at certain heights, which we will denote by the functions τ with the corresponding indices. Let $x \in W^n_{i,k} \cap \sigma$ and suppose that we have already cut the cylinder $\hat{W}^n_{i,k}$ at some constant height $\tau_{i,k} \in [0, 1/2]$ and chosen $\hat{W}^n_{j,l}$ to be its pair. Then the holonomy map between these components gives the point $h(x) \in W^n_{j,l}$. Observe that in order to have a measure preserving bijection, the height $\tau_{j,l}(h(x))$ where we have to cut the second cylinder $\hat{W}^n_{j,l}$ is then completely determined in all the points $h(x) \in W^n_{j,l} \cap \sigma$ (but not yet on the whole component $W^n_{j,l}$) by the equation

$$\tau_{i,k} \cdot \rho^{(n)}_{i,k}(x)dx = \tau_{j,l}(h(x)) \cdot \rho^{(n)}_{j,l}(h(x)) \cdot (\mathcal{J}_{W^n_{i,k}} h)(x)dx, \tag{5.9}$$

where $x \in W^n_{i,k} \cap \sigma$ and $\rho^{(n)}_{i,k}$ is the density of the n-th iterate of the measure ν_i, i.e.

$$\rho^{(n)}_{i,k}(F^n(y)) \cdot (\mathcal{J}_{W_i} F^n)(y)dy = \rho_i(y)dy, \qquad \text{where } y \in F^{-n} W^n_{i,k}$$

(and similar notation holds for the indices $\{j, l\}$). Remember that we have to guarantee that $\tau_{j,l}(h(x)) \in [0, 1]$. Using (2.3) and the definition of standard pairs (Definition 2.4) we know that for all $x_1, x_2 \in W^n_{i,k}$

$$\begin{aligned}
&|\ln \rho^{(n)}_{i,k}(x_1) - \ln \rho^{(n)}_{i,k}(x_2)| \\
&= |\ln[\rho_i(F^{-n}x_1)(\mathcal{J}_{W^n_{i,k}} F^{-n})(x_1)] - \ln[\rho_i(F^{-n}x_2)(\mathcal{J}_{W^n_{i,k}} F^{-n})(x_2)]| \\
&\leq |\ln \rho_i(F^{-n}x_1) - \ln \rho_i(F^{-n}x_2)| + |\ln(\mathcal{J}_{W^n_{i,k}} F^{-n})(x_1) - \ln(\mathcal{J}_{W^n_{i,k}} F^{-n})(x_2)| \\
&\leq C_r \theta^{s_+(F^{-n}x_1, F^{-n}x_2)} + C' d_{W_i}(x_1, x_2).
\end{aligned} \tag{5.10}$$

Hence by choosing n sufficiently large and the diameter of our magnet (determined by the value of δ_1) small enough we can even ensure that the oscillation

of the densities on the connected components crossing σ is very small, say for every $x, y \in W^n_{i,k} \cap \sigma$ we have that $\frac{\rho^{(n)}_{i,k}(x)}{\rho^{(n)}_{i,k}(y)} \in [1-\varepsilon, 1+\varepsilon]$.
Similarly, by appropriately choosing the parameters of our magnet, we can ensure that the Jacobian of the holonomy map (from $W^n_{i,k}$ to $W^n_{j,l}$) is very close to 1 at every point x of the intersection $W^n_{i,k} \cap \sigma$. Indeed, recall that in Lemma 4.5 we showed that for the holonomy map $h : W_1 \to W_2$ between two u-manifolds we have the estimate

$$|\ln(\mathcal{J}_{W_1} h)(x)| \leq \tilde{C} \cdot d(x, h(x))^a + \hat{C} \cdot \delta,$$

where $\delta > 0$ is a quantity that measures how parallel the tangent spaces of the u-manifolds are. We apply this result with the choice $W_1 = W^n_{i,k}$ and $W_2 = W^u$, where W^u is the unstable manifold on which the magnet σ is built on. The u-manifold $W^n_{i,k}$ crosses σ, actually there is a ball of radius $\delta_1/3 + \delta_3$ in it that contains all the intersections with the magnet. We will denote the center of this ball by c and the ball itself by $B_{i,k}(c, \delta_1/3 + \delta_3)$ in the next lines. Due to our constructions and (3.4) $B_{i,k}(c, \delta_1/3 + \delta_3)$ is overshadowed by W^u with s-distance less than $4\left(\frac{2.1}{C_t} + 1\right) \cdot \delta_3 =: C_0 \cdot \delta_3$. Hence, for the points $x \in W^n_{i,k} \cap \sigma$ and $h(x) \in W^u$, the inequality $d(x, h(x)) \leq C_0 \cdot \delta_3$ holds. Now we will argue that δ can be made arbitrarily small by choosing the ratio δ_3/δ_1 small enough. Recall that δ was defined in the following way. Take an arbitrary unit vector $\underline{e} \in T_x W^n_{i,k}$, consider its parallel translate to $h(x) \in W^u$ and decompose it as $\underline{e} = \underline{v} + \underline{s} \in T_{h(x)} W^u \oplus E^s_{h(x)}$. Then $\delta := \max_{\underline{e}}\{\|\underline{s}\|\}$. Assume that $\delta > 6C_0\delta_3/\delta_1$ holds, we show that this leads to a contradiction. Let S be the s-manifold through c with constant tangent space parallel to $E^s_{h(x)}$. This intersects W^u in a point d. Due to the uniform bound on the curvature of u-manifolds $W^n_{i,k}$ is very close to be a d_u-dimensional hyperplane. So we can find a unit vector $\underline{e}$ at $T_c W^n_{i,k}$ with decomposition $\underline{e} = \underline{w} + \underline{s} \in T_d W^u \oplus T_d S$ and $\|\underline{s}\| \geq \delta/2$. Then we move along this direction $\delta_1/3$ distance towards the side of $B_{i,k}(c, \delta_1/3 + \delta_3)$. This results in a stable component of length at least $\delta_1/3 \cdot \delta/2$. According to our assumption on δ this is more than $C_0 \cdot \delta_3$, which is a contradiction since $B_{i,k}(c, \delta_1/3 + \delta_3)$ is overshadowed by W^u with s-distance less than $C_0 \cdot \delta_3$. Therefore $\delta \leq 6C_0\delta_3/\delta_1$ must hold.
We conclude that for the holonomy map $h_1 : W^n_{i,k} \to W^u$ we have

$$|\ln(\mathcal{J}_{W^n_{i,k}} h_1)(x)| \leq \tilde{C} \cdot (C_0\delta_3)^a + \hat{C} \cdot 6C_0\delta_3/\delta_1.$$

The same argument holds for the holonomy $h_2 : W^n_{j,l} \to W^u$. Finally observe that the Jacobian of the holonomy map between the initial u-manifolds, i.e. from $W^n_{i,k}$ to $W^n_{j,l}$ is the product of the Jacobians of the maps h_1 and h_2^{-1}. Hence for the map $h : W^n_{i,k} \to W^n_{j,l}$

$$|\ln(\mathcal{J}_{W^n_{i,k}} h)(x)| \leq 2\tilde{C} \cdot (C_0\delta_3)^a + 2\hat{C} \cdot 6C_0\delta_3/\delta_1, \tag{5.11}$$

which is clearly close to zero if both δ_3 and the ratio δ_3/δ_1 is close to zero.
This implies that $(\mathcal{J}_{W^n_{i,k}} h)(x) \in [1-\varepsilon, 1+\varepsilon]$ for some very small $\varepsilon > 0$. Hence it is enough to make a measure preserving bijection between our cylinders, with $\tau_{j,l} \in [0, 1/2]$ "cutting heights" on $W^n_{j,l}$'s, assuming that the densities on these cylinders are actually constant and the Jacobian of the holonomy map is equal

to one. Because then according to our observations the multiplicative error we make is between $1-\varepsilon_0$ and $1+\varepsilon_0$, so keeping the resulting heights $\tau_{i,k}$ constant, the relation (5.9) gives us the desired height functions $\tau_{j,l}(h(x))$ (just on $W^n_{j,l} \cap \sigma$ at the moment), which will take values in say $[0, 0.6]$.

!!!Inner Comment!!! 5.12. *A következő lemma talán kihagyható, túl részletesen magyaráz. Más tészta, hogy ennek kapcsán felmerül a jó öreg kérdés a mérhetőségről...*

Lemma 5.13. *Let $A_1, A_2, \dots$ and $B_1, B_2, \dots$ be two countable families of disjoint cylinders with the same height 1. Denote their volumes by $m(A_i)$ and $m(B_j)$ respectively and assume that $d_0 \le \sum_{i=1}^{\infty} m(A_i) \le 1$ and $d_0 \le \sum_{i=1}^{\infty} m(B_j) \le 1$ for some positive constant d_0. Then for every positive number $d < d_0$ there exist constants $\tau_{i,k}$ and $\hat{\tau}_{j,l}$ and a bijection $b : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ such that*

- $\tau_{i,k}, \hat{\tau}_{j,l} \in [0, 1/2]$ *for all (i,k) and (j,l) in $\mathbb{N}\times\mathbb{N}$,*
- $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \tau_{i,k}\cdot m(A_i) = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} \hat{\tau}_{j,l}\cdot m(B_j) = \frac{d}{2}$,
- *for all $(i,k) \in \mathbb{N}\times\mathbb{N}$ we have $\tau_{i,k}\cdot m(A_i) = \hat{\tau}_{b(i,k)}\cdot m(B_{b(i,k)_1})$,*

where $b(i,k)_1$ denotes the first coordinate of $b(i,k)$.

Proof. Given d fixed we look for the first i_0 and j_0 such that the total volumes of the corresponding family of cylinders up to these indices just exceed or reach d. This way we reduce our families to consist only of finitely many rectangles. Now we construct the constants $\tau_{i,k}$ and $\hat{\tau}_{j,l}$ in an algorithmic way. First pick A_1 and B_1. If $m(A_1) = m(B_1)$ then let $\tau_{1,1} = \hat{\tau}_{1,1} = \frac{1}{2}$, define $b(1,1) = (1,1)$ and throw away both cylinders. If the volumes are not equal, say $m(A_1) < m(B_1)$ (the other case is analogous) then cut down the half of A_1 by defining $\tau_{1,1} = \frac{1}{2}$ and cut down a cylinder from B_1 with the same volume, i.e. let $\hat{\tau}_{1,1} = \frac{m(A_1)}{2m(B_1)}$. Finally define $b(1,1) = (1,1)$ and throw away the cylinder A_1 and the cylinder that is the part of B_1 which is just cut down. Now compare the volume of the half of A_2 to the volume of what remained from the half of B_1 after removing the previous cylinder.

- If they are equal, then we cut down both of them by setting $\tau_{2,1} = \frac{1}{2}$ and $\hat{\tau}_{1,2} = \frac{1}{2} - \hat{\tau}_{1,1}$. Define $b(2,1) = (1,2)$ and throw away both of the remaining cylinders,
- if $(\frac{1}{2} - \hat{\tau}_{1,1})m(B_1) > \frac{1}{2}m(A_2)$, then we cut down the half of A_2 and a cylinder of the same size from the remainder of B_1, i.e. let $\tau_{2,1} = \frac{1}{2}$, $\hat{\tau}_{1,2} = \frac{m(A_2)}{2m(B_1)}$. Define $b(2,1) = (1,2)$ and this time we throw away only A_2,
- if $(\frac{1}{2} - \hat{\tau}_{1,1})m(B_1) < \frac{1}{2}m(A_2)$, then we cut down a cylinder from B_1 in a way that what remains will have volume $\frac{1}{2}m(B_1)$ and a cylinder of the same size from A_2, i.e. let $\tau_{2,1} = \frac{(\frac{1}{2}-\hat{\tau}_{1,1})m(B_1)}{m(A_2)}$, $\hat{\tau}_{1,2} = \frac{1}{2} - \hat{\tau}_{1,1}$. Define $b(2,1) = (1,2)$ and throw away B_1.

We proceed in an analogous manner. By now it should be clear that the index (i,k) corresponds to the k'th cylinder cut down from A_i, which has volume $\tau_{i,k} \cdot m(A_i)$ and the notation is similar for the index (j,l). Now assume that the cylinders being currently in hand are A_i and B_j and we have already cut down k_0-1 and l_0-1 subcylinders from them respectively (in which case either k_0 or l_0 or both is equal to 1). There are three cases but in either way define $b(i,k_0)=(j,l_0)$. The possibilities are

- $(\frac{1}{2}-\sum_{k=1}^{k_0-1}\tau_{i,k})m(A_i)=(\frac{1}{2}-\sum_{l=1}^{l_0-1}\hat{\tau}_{j,l})m(B_j)$. In this case let $\tau_{i,k_0}=\frac{1}{2}-\sum_{k=1}^{k_0-1}\tau_{i,k}$ and $\hat{\tau}_{j,l_0}=\frac{1}{2}-\sum_{l=1}^{l_0-1}\hat{\tau}_{j,l}$, throw away both A_i and B_j and replace i by $i+1$, j by $j+1$, k_0 by 1 and also l_0 by 1,
- $(\frac{1}{2}-\sum_{k=1}^{k_0-1}\tau_{i,k})m(A_i)<(\frac{1}{2}-\sum_{l=1}^{l_0-1}\hat{\tau}_{j,l})m(B_j)$. If this is the case then let $\tau_{i,k_0}=\frac{1}{2}-\sum_{k=1}^{k_0-1}\tau_{i,k}$ and $\hat{\tau}_{j,l_0}=\frac{\tau_{i,k_0}\cdot m(A_i)}{m(B_j)}$, throw away the cylinder A_i and replace i by $i+1$, k_0 by 1 and l_0 by l_0+1 (no modification of j is needed),
- $(\frac{1}{2}-\sum_{k=1}^{k_0-1}\tau_{i,k})m(A_i)>(\frac{1}{2}-\sum_{l=1}^{l_0-1}\hat{\tau}_{j,l})m(B_j)$. In such a case let $\hat{\tau}_{j,l_0}=\frac{1}{2}-\sum_{l=1}^{l_0-1}\hat{\tau}_{j,l}$ and $\tau_{i,k_0}=\frac{\hat{\tau}_{j,l_0}\cdot m(B_j)}{m(A_i)}$, throw away the cylinder B_j and replace j by $j+1$, k_0 by k_0+1 and l_0 by 1 (no modification of i is needed).

Running this procedure at each step at least one cylinder is thrown away, so within finitely many steps the algorithm terminates. Observe that a cylinder was thrown away exactly after half of its volume was cut down, so at the end of the process we coupled half of the volume of one finite family of cylinders, which still exceeds $\frac{d}{2}$. Hence multiplying all $\tau_{i,k}$'s and $\hat{\tau}_{j,l}$'s with a constant (which is less than 1) we can ensure that the total coupled volume is exactly $\frac{d}{2}$ and multiplication by a constant does not harm the first and the third condition for the $\tau_{i,k}$'s and $\hat{\tau}_{j,l}$'s in the lemma. To be absolute precise for all indices (i,k) and (j,l) that we have never used so far (note that there are infinitely many) we fix an arbitrary extension of the bijection b and define all $\tau_{i,k}$'s and $\hat{\tau}_{j,l}$'s to be zero. $\square$

Apply Lemma 5.13 to our cylinders crossing the magnet with $m(A_{i,k})=\int_{W^n_{i,k}\cap\sigma}\rho^{(n)}_{i,k}(x)\,dx$ and $m(B_{j,l})=\int_{W^n_{j,l}\cap\sigma}\rho^{(n)}_{j,l}(x)\,dx$. The resulting b will be the bijection between our cylinders and keeping the constant heights $\tau_{i,k}\in[0,1/2]$ the relation (5.9) gives us the corresponding heights $\hat{\tau}_{j,l}(h(x))\in[0,0.6]$. We again emphasize that $\hat{\tau}_{j,l}(h(x))$ is defined only on the set $W^n_{j,l}\cap\sigma$ so far. But since ρ_i, ρ_j are by definition dynamically Hölder continuous and so is $(\mathcal{J}_{W^n_{i,k}}h)(x)$ by Proposition 4.7 and as a nature of hyperbolic systems $s_+(x_1,x_2)=s_+(h(x_1),h(x_2))$ it follows that $\hat{\tau}_{j,l}(h(x))$ is also dynamically Hölder continuous on its present domain. As a final step of the coupling of measures we want to ensure that we proceed with the same kind of objects as we had before. When a component of an iterate of a standard family crossed the magnet we considered the corresponding cylinder and removed certain "spikes" with various heights from it, so it no longer remained a cylinder based on a u-manifold (equipped with a regular density). To get cylinders we cut this object into two by defining a cutting surface fitting on the "spikes". We make this argument precise below.

Lemma 5.14. *Let W be a u-manifold and $\sigma \subset W^1_\infty$ be a closed subset for a δ-filtration of W (recall Definition 3.2). Suppose that f is a function defined on σ in a way that it is dynamically Hölder continuous and takes values in the interval $[0,1]$. We claim that there exists an extension $\tilde{f}$ of f such that $\tilde{f}$ is defined almost everywhere on W (and at every point of σ of course), takes values in the interval $[0,1]$ as well and it is dynamically Hölder continuous on its domain.*

For uniformly hyperbolic systems with one dimensional unstable manifolds (or just unstable curves) an independent proof of this lemma is given in [14, Lemma 15]. Our argument here works also for higher dimensional u-manifolds.

Proof. Note that for two points x and y the equation $s_+(x,y) = \infty$ implies that they are on the same local stable manifold. This can not happen with the points of W, hence the set σ is totally disconnected and for every $x, y \in \sigma$ we have $s_+(x,y) < \infty$, but this quantity is not necessarily bounded. Recall that $\xi^s_n(x)$ denotes the connected component of $\mathcal{M} \setminus \mathcal{S}^{(n)}$ containing x. Since $\sigma \cap \partial \xi^s_n(x) = \emptyset$ it follows that $\sigma \cap \xi^s_n(x)$ is a closed set for every $x \in W$ and $n \geq 0$.
Now consider a sequence $x_k \in \sigma$, such that $x_k \to x \in \sigma$ as $k \to \infty$. Choose a subsequence x_{k_n} in a way that for all n the inequality $d(x, x_{k_n}) < \delta_2 \Lambda^{-n}$ holds. By construction of a δ_2-filtration it is clear that $\forall n \geq 0 : B(x, \delta_2 \Lambda^{-n}) \cap \mathcal{S}^{(n)} = \emptyset$, where $B(x,r)$ denotes the ball in $\mathcal{M}$ centered at x with radius r. Hence $s_+(x, x_{k_n}) > n$ and so the dynamical Hölder continuity of f on σ implies that $|\ln f(x) - \ln f(x_{k_n})| < C \cdot \theta^{-n}$. This holds for every $n \geq 0$ so $\lim\limits_{n\to\infty} f(x_{k_n}) = f(x)$, showing that f is continuous on σ. Therefore it attains its minimum on each set of the form $\sigma \cap \xi^s_n(x)$. Now we construct the extension of f as a limit of a recursion. Let $f_0(x) : W \to \mathbb{R}$ be the constant function defined as $f_0(x) = \min\limits_{y \in \sigma} f(y)$. If $f_k(x)$ is already defined for some $k \in \mathbb{N}$ then $f_{k+1}(x)$ will be as follows:

- if $\sigma \cap \xi^s_{k+1}(x) = \emptyset$ then let $f_{k+1}(x) = f_k(x)$,
- if $\sigma \cap \xi^s_{k+1}(x) \neq \emptyset$ then let $f_{k+1}(x) = \min\limits_{y \in \sigma \cap \xi^s_{k+1}(x)} f(y)$.

Finally let $\tilde{f} : W \to \mathbb{R}$ be $\lim\limits_{n\to\infty} f_n$. We claim that $\tilde{f}$ is the extension we are looking for. It is clear that for all $x \in W$ and $k \in \mathbb{N}$ the function $f_k(x)$ is constant on each connected component of $W \cap \xi^s_k(x)$, $f_k(x) \leq 1$ and $f_k(x) \leq f_{k+1}(x)$. Hence it follows that $\tilde{f}$ is well defined on $W \setminus \mathcal{S}^{(\infty)}$, i.e. almost everywhere on W. It is indeed an extension of f, because if $x \in \sigma$ then by the monotonity of the sequence $f_k(x)$ it follows that $\tilde{f}(x) \leq f(x)$. Now if it would be strictly less, then it would mean for every $k \geq 0$ there exists a point $y_k \in \sigma \cap \xi^s_k(x)$ different from x, where f attains its minimum in the k-th step of the construction. Clearly

$s_+(x, y_k) > k$, hence

$$\begin{aligned} |\ln \tilde{f}(x) - \ln f(x)| \leq \\ & |\ln \tilde{f}(x) - \ln f_k(x)| + |\ln f_k(x) - \ln f(x)| = \\ & = |\ln \tilde{f}(x) - \ln f_k(x)| + |\ln f(y_k) - \ln f(x)| < \\ & < \varepsilon + C \cdot \theta^{s_+(x,y_k)} \leq \varepsilon + C \cdot \theta^k < \varepsilon_0 \end{aligned}$$

for any ε_0 by chosing k large enough and using the definition of $\tilde{f}(x)$ and the dynamical Hölder continuity of f on the set σ. But this implies that $\tilde{f}(x)$ can not be less than $f(x)$ so it is indeed an extension. What remains is to prove that $\tilde{f}$ is dynamically Hölder continuous on its whole domain. Let $x, y \in W \setminus \mathcal{S}^{(\infty)}$, there are two cases left:

if $x \in \sigma$ but $y \notin \sigma$ (the reversed case is analogous) then there is a largest index k_0 such that $\sigma \cap \xi^s_{k_0}(y) \neq \emptyset$. Let us denote by z the point where the minimum of f is attained on this set. Observe that then $\tilde{f}(y) = f(z)$ and since $x \in \sigma$ it is clear that $k_0 \geq s_+(x, y) - 1$. So $z \in \sigma \cap \xi^s_{s_+(x,y)-1}(x)$ and by definition $\xi^s_{s_+(x,y)-1}(x) = \xi^s_{s_+(x,y)-1}(y)$, hence $s_+(x, z) \geq s_+(x, y)$. Therefore

$$|\ln \tilde{f}(x) - \ln \tilde{f}(y)| = |\ln f(x) - \ln f(z)| \leq C \cdot \theta^{s_+(x,z)} \leq C \cdot \theta^{s_+(x,y)}$$

If $x, y \notin \sigma$ then there are maximal indices k and l such that $\sigma \cap \xi^s_k(x) \neq \emptyset$ and $\sigma \cap \xi^s_l(y) \neq \emptyset$. Then

- either both k and l are less than $s_+(x, y)$ in which case they are actually equal and so are the values of $\tilde{f}(x)$ and $\tilde{f}(y)$ by construction, so there is no problem with regularity,

- or at least one of the indices k, l is at least $s_+(x, y)$. Then let $w \in \sigma \cap \xi^s_k(x)$ and $z \in \sigma \cap \xi^s_l(y)$ be the points where the minimum of f is attained on these sets respectively (it may happen that $w = z$ actually, but in this case $\tilde{f}(x) = \tilde{f}(y)$ as well so again no problem with regularity). By construction $\tilde{f}(x) = f(w)$ and $\tilde{f}(y) = f(z)$. Observe that $w \in \sigma \cap \xi^s_n(x)$ for all $n \leq k$ and $z \in \sigma \cap \xi^s_m(y)$ for all $m \leq l$, hence $s_+(x, y) \leq s_+(w, z)$. Therefore

$$|\ln \tilde{f}(x) - \ln \tilde{f}(y)| = |\ln f(w) - \ln f(z)| \leq C \cdot \theta^{s_+(w,z)} \leq C \cdot \theta^{s_+(x,y)}.$$

□

Consider a cylinder that crosses the magnet σ. If it belongs to the first family then it was denoted by $\hat{W}^n_{i,k}$ for some index i. For simplicity we neglect the index k used to mark the components of $F^n(W_i)$ and we will just simply refer to these components as W^n_i. Lemma 5.13 produced finitely many nonzero constant cutting heights $\tau_{i,k}$ (this k is now different from the one we have just dropped) on the set $W^n_i \cap \sigma$, such that $\mathcal{T}_i := \sum_{k=1}^{\infty} \tau_{i,k}$ is in $[0, 1/2]$. The points that are coupled now are of the form (x, t), where $x \in W^n_i \cap \sigma$ and $t \in [0, \mathcal{T}_i]$. We now remove these from $\hat{W}^n_i$. The constant function $\mathcal{T}_i$ can obviously be extended from $W^n_i \cap \sigma$ to the set W^n_i by keeping it constant, so it will cut our original cylinder $\hat{W}^n_i$ into two parts. Observe that even after the removal of the coupled points there are actually two cylinders!

- One is the upper part containing no coupled points. It is based on the u-manifold W_i^n and has (constant) height $1-\mathcal{T}_i$.
- The lower part contained all the coupled points and after removing them it will still be a cylinder, but based on the u-manifold $W_i^n \setminus \sigma$ with (constant) height $\mathcal{T}_i$.

Both of these cylinders carry the measure with density $\rho_i^{(n)}(x)\,dx\,dt$. Now with an appropriate linear transformation in the t-coordinate we strech both of them to recover their unit height. This will modify the densities in a way, that they will be $(1-\mathcal{T}_i)\rho_i^{(n)}(x)\,dx\,dt$ and $\mathcal{T}_i\rho_i^{(n)}(x)\,dx\,dt$ respectively.
Note that multiplication by a constant does not change the dynamical Hölder continuity of these functions. Hence, apart from the fact that the measures are not normalized (yet, which is still just a multiplication by a constant) the upper parts remaining from the first family after the coupling are standard pairs. This is however not true for the lower parts, since the base u-manifolds $W_i^n \setminus \sigma$ are not admissible (their boundaries are not piecewise smooth). We will have to iterate these lower parts further and wait for some moments, when some of the components of their image will be standard pairs again. This will be an important part of the next subsection.
For the second family we do exactly the same steps (we also neglect the index l from now on). The only difference is that the cutting heights $\hat{\tau}_{j,l}(h(x))$ (this l is again different from the one we have just dropped) given by Lemma 5.13 and (5.9) are no longer constants, but still dynamically Hölder continuous on the set $W_j^n \cap \sigma$. Then it is easy to see that the function $\hat{\mathcal{T}}_j(h(x)) := \sum_{l=1}^{\infty} \tau_{j,l}(h(x))$ is also dynamically Hölder continuous (and so is $1-\hat{\mathcal{T}}_j(h(x))$, since $\hat{\mathcal{T}}_j(h(x))$ is bounded away from 1) so by Lemma 5.14 we can extend it to the set W_j^n (or to just a full measure set in it to be precise). This will cut the original cylinder $\hat{W}_j^n$ again into two parts, which we treat exactly the same way as before.
Even though the product of two dynamically Hölder continuous functions still has this kind of regularity, the upper parts after the cutting are not exactly standard pairs again but here not just because of the lack of normalization. The real problem with them is that the constant C_0 for which

$$|\ln \hat{\mathcal{T}}_j(x)\rho_j^{(n)}(x) - \ln \hat{\mathcal{T}}_j(y)\rho_j^{(n)}(y)| \le C_0 \cdot \theta^{s_+(x,y)}$$

holds, is maybe larger than the constant C_r in the definition of standard pairs. However if $\hat{\mathcal{T}}_j(y)$ is dynamically Hölder continuous with a constant say C_1, then $C_0 \le C_r + C_1$. Due to this uniform bound these measures will regularize in a fixed number of iterations in the same way as described in Remark 3.13. So not immediately after the coupling, but just after a fixed number of iterations the upper parts of the second family after the cutting will be standard pairs.
The argument for the densities $(1-\hat{\mathcal{T}}_j(y))\rho_j^{(n)}(y)$ on the lower parts is the same. But even though these densities regularize in a fixed number of iterations, the lower parts will not be standard pairs at this stage. The problem is the same as it is for the first family: the base u-manifolds $W_j^n \setminus \sigma$ are not admissible.

5.3 Coupling time

In the previous subsection we described how to couple measures on standard families, when they are getting close to each other on the magnet. By our construction it is clear that the first part of the Coupling Lemma (Lemma 5.2) holds whatever the coupling time $\Gamma(x,t)$ is. In this subsection we define the coupling time for arbitrary standard families and prove that it admits the uniform exponential tail bound in the second part of Lemma 5.2.
Consider two proper standard families $\mathcal{G} = \{(W_i, \nu_i)\}_{i\in I}$ and $\mathcal{E} = \{(W_j, \nu_j)\}_{j\in J}$. Corollary 5.9 implies that after n_0 iterations their images will cross the special magnet σ and the preimage of the intersections have $\mu_{\mathcal{G}}$ (or $\mu_{\mathcal{E}}$) measure at least d_0 in $\mathcal{G}$ (or in $\mathcal{E}$ respectively). At this stage we apply our coupling procedure described in the previous subsection to couple $\frac{d_0}{2}$ amount of the measures. The coupled points in the cylindrical extensions of W_i's are of the form $\hat{W}_{1,c} := \{(x,t) \in \hat{\mathcal{G}} | F^{n_0}(x) \in W_i^{n_0} \cap \sigma$ and $t \in [0, \mathcal{T}_i(x)]$ for some index i such that $W_i^{n_0}$ crosses $\sigma\}$. We set the coupling time $\Gamma(x,t) = n_0$ for every $(x,t) \in \hat{W}_{1,c}$ (here the index 1 refers to the first family $\mathcal{G}$ and c to the word coupling).

!!!Inner Comment!!! 5.15. *Ezekben a sorokban n_0 helyett, mindig valami $n_0 + k$-nak kell majd szerepelnie, ahol $0 \le k < n_0$, de a konkrét értéke az adott standard pároktól függ.*

After this coupling step we are left with components of three different kinds basically. There are components of $F^{n_0}W_i$'s that do not cross the magnet and some which do. The cylindrical extensions of the later are divided into two objects, which we described briefly in the final lines of the previous subsection. Here we treat them in a slightly different way and in more detail. Let $W_i^{n_0}$ be a component of $F^{n_0}(\mathcal{G})$ that crosses the magnet. Then as we have seen in the proof of Lemma 5.7 there is a ball $B_i \subset W_i^{n_0}$ with radius $\delta_1/3 + 2\delta_3$, which contains all points of the intersection $W_i^{n_0} \cap \sigma$.

!!!Inner Comment!!! 5.16. *Lemma 5.7 nem biztos, hogy létezni fog a változtatások után.*

We cut out this ball from $W_i^{n_0}$ and consider the cylindrical extension of it. During the coupling some sticks have been removed from this extension with heights $\mathcal{T}_i(x)$ altogether forming the set $\hat{W}_{1,c}$. This function is defined by the coupling only on $B_i \cap \sigma$, but with the help of Lemma 5.14 we can extend it to B_i and so we can cut the cylinder $B_i \times [0,1]$ into two pieces. We will refer to the lower part as B_i^1 and the upper part as B_i^2.
To proceed with the coupling of measures we need to guarantee that the non-coupled parts will return to the magnet and again cross it with an intersection of fix relative measure. By Corollary 5.9 it is enough to show that after some iterations the non-coupled parts will be *proper* standard families.
Considering the standard family that consists of standard pairs not affected by the coupling: Observe that those components that do not cross the magnet and the objects $W_i^{n_0} \setminus B_i$ are actually standard pairs (after normalizing the measures on them). This almost holds for the B_i^2's too, just a fixed number of iterations is needed for the measures to regularize on them (cf. Remark 3.13). Actually this number is uniformly bounded and may even reduced to be 1 by choosing the constant C_r in Definition 2.4 large enough. Now we

show that the union of all these three type of objects will form a *proper* standard family within a fixed number of iterations.
Let us index the components of $F^{n_0}(\mathcal{G})$ by $i \in \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$, where $\mathcal{I}_1$ is for components which cross the magnet and $\mathcal{I}_2$ is for those which do not. We double every index $i \in \mathcal{I}_1$ and use $(i,1)$ and $(i,2)$ to refer to $W_i^{n_0} \setminus B_i$ and B_i^2 (the upper part of B_i after cutting it with $\mathcal{T}_i(x)$) respectively. These objects are equipped with probability measures induced by the original measure $\mu_{\mathcal{G}}$ on $\mathcal{G}$, which are as follows.

- If $i \in \mathcal{I}_2$ then the probability measure on $W_i^{n_0}$ has density

$$d\nu_i^{(n_0)}(x) = \rho_i^{(n_0)}(x) = \frac{\rho(F^{-n_0}(x))(\mathcal{J}_{W_i^{n_0}} F^{-n_0})(x)}{\int\limits_{W_i^{n_0}} \rho(F^{-n_0}(y))(\mathcal{J}_{W_i^{n_0}} F^{-n_0})(y)\, dy}$$

- The situation is almost the same for every set $W_i^{n_0} \setminus B_i$, $i \in \mathcal{I}_1$. The only difference is that the density of $\nu_{(i,1)}$ is supported on $W_i^{n_0} \setminus B_i$ so this has to be the domain of integration in the previous formula.

- The case of B_i^2's is somewhat different however. Here the density changes since we kept only a fraction of the points from B_i after the coupling. What remains is

$$d\nu_{(i,2)}^{(n_0)}(x) = \rho_{(i,2)}^{(n_0)}(x) = \frac{\rho(F^{-n_0}(x))(\mathcal{J}_{B_i} F^{-n_0})(x)(1-\mathcal{T}_i(x))}{\int\limits_{B_i} \rho(F^{-n_0}(y))(\mathcal{J}_{B_i} F^{-n_0})(y)(1-\mathcal{T}_i(y))\, dy}$$

We need to adapt a factor measure on the new index set (the indices $(i,1),(i,2)$ are in count!) in a way that it preserves the original weight of each component. So we take these weights, which are (in the order of the previous itemization)

- $c_i^{n_0} = \int\limits_{W_i^{n_0}} \rho(F^{-n_0}(x))(\mathcal{J}_{W_i^{n_0}} F^{-n_0})(x)\, dx$, for $i \in \mathcal{I}_2$,

- $c_{(i,1)}^{n_0} = \int\limits_{W_i^{n_0} \setminus B_i} \rho(F^{-n_0}(x))(\mathcal{J}_{W_i^{n_0} \setminus B_i} F^{-n_0})(x)\, dx$, for $(i,1)$ which is one copy of the index $i \in \mathcal{I}_1$,

- $c_{(i,2)}^{n_0} = \int\limits_{B_i} \rho(F^{-n_0}(x))(\mathcal{J}_{B_i} F^{-n_0})(x)(1-\mathcal{T}_i(x))\, dx$ for $(i,2)$ which is the other copy of the index $i \in \mathcal{I}_1$.

Since we removed some mass from the standard family (corresponding to B_i^1's) we have to normalize these weights to get the new (probability) factor measure, which we will denote by $\tilde{c}_j^{n_0}$ for any kind of index j. We constructed the cutting height function in a way that $\mathcal{T}_i(x) \in [d_0/2, 3/5]$ and from this it is easy to verify that

$$\sum_{i \in \mathcal{I}_1} c_{(i,1)}^{n_0} + c_{(i,2)}^{n_0} + \sum_{i \in \mathcal{I}_2} c_i^{n_0} \geq \frac{2}{5}$$

and hence for any kind of index j we have

$$\tilde{c}_j^{n_0} \leq \frac{5}{2} c_j^{n_0}. \tag{5.12}$$

Now we give an estimate on how much the cutting of B_i from $W_i^{n_0}$ increases the measure of the ε-neighborhood of the boundary of the resulting set compared to the ε-neighborhood of the boundary of the original set $W_i^{n_0}$. For a fixed $i \in \mathcal{I}_1$ we introduce the sets

$$\mathcal{N}_{B_i^{in}} = \{x \in B_i | d(x, \partial B_i) < \varepsilon\} \quad \mathcal{N}_{B_i^{out}} = \{x \in W_i^{n_0} \setminus B_i | d(x, \partial B_i) < \varepsilon\}$$
$$\mathcal{N}_{W_i^{n_0}} = \{x \in W_i^{n_0} | d(x, \partial W_i^{n_0}) < \varepsilon\}$$

With these notations the measure we want to estimate for every ε is

$$\nu_i^{(n_0)}(\{x \in W_i^{n_0} | d(x, \partial W_i^{n_0} \cup \partial B_i) < \varepsilon\}) = \nu_i^{(n_0)}(\mathcal{N}_{W_i^{n_0}} \cup \mathcal{N}_{B_i^{in}} \cup \mathcal{N}_{B_i^{out}})$$

Using that $W_i^{n_0}$ has bounded curvature and its diameter is at most δ_0 plus the fact that $\nu_i^{(n_0)}$ is uniformly equivalent to the normalized Lebesgue measure on $W_i^{n_0}$ a geometrical argument, similar to the one in [6, Lemma 4.2] or [5, Appendix C] but using polar coordinates instead of cartesian ones, shows that

$$\nu_i^{(n_0)}(\mathcal{N}_{B_i^{in}}) \leq const \cdot \nu_i^{(n_0)}(\mathcal{N}_{W_i^{n_0}}) \quad \nu_i^{(n_0)}(\mathcal{N}_{B_i^{out}}) \leq const \cdot \nu_i^{(n_0)}(\mathcal{N}_{W_i^{n_0}})$$

with uniform constants and hence there is another uniform constant $C > 0$ for which

$$\nu_i^{(n_0)}(\{x \in W_i^{n_0} | d(x, \partial W_i^{n_0} \cup \partial B_i) < \varepsilon\}) \leq C \cdot \nu_i^{(n_0)}(\mathcal{N}_{W_i^{n_0}}) \tag{5.13}$$

We denote by $\tilde{\mathcal{G}}_{n_0}$ the standard family consisting of the non-coupled parts, more precisely of u-manifolds from $\cup_{i \in \mathcal{I}_1}(W_i^{n_0} \setminus B_i \cup B_i^2) \bigcup (\cup_{i \in \mathcal{I}_2} W_i^{n_0})$ and the induced measures on them. The standard family before the coupling is denoted by $\mathcal{G}_{n_0}$. The measures induced by these families are $\mu_{\tilde{\mathcal{G}}_{n_0}}$ and $\mu_{\mathcal{G}_{n_0}}$ respectively. Using these notations, (5.12) together with (5.13) implies that

$$\mathcal{Z}_{\tilde{\mathcal{G}}_{n_0}} \leq \frac{5}{2} C \mathcal{Z}_{\mathcal{G}_{n_0}}. \tag{5.14}$$

But at the moment of coupling we had $\mathcal{Z}_{\mathcal{G}_{n_0}} \leq c_3$ and so by Corollary 3.16 and Corollary 5.9 it follows that after a fixed number of iterations a fixed proportion of the standard family formed by the non coupled parts will be on the magnet and so it will be ready for coupling.

Considering the lower parts B_i^1 after the coupling: As we have already pointed out the main problem with the set $B_i^1 \setminus \sigma$ is that it is not a standard pair, because its boundary is not piecewise smooth. We will deal with this issue in the rest of this subsection.

Similar to what we did with B_i^2's before, we stretch the set B_i^1 in the t-direction with an appropriate linear transformation to get a cylinder with unit height. Therefore we will have $(B_i, \tilde{\nu}_i)$ as an initial object, where B_i is a ball (in its inner metric) with radius $\delta_1/3 + 2\delta_3$ and $\tilde{\nu}_i$ is a measure with density $\frac{1}{c}\rho_i^{(n_0)}(x) \cdot \mathcal{T}_i(x)$, where c is just the normalizing constant. Note that even $(B_i, \tilde{\nu}_i)$ itself is not a standard pair, since the measure is not regular in the sense of Definition 2.4. It is however dynamicaly Hölder continuous with regularity constant $C_r + C_0$ (instead of just C_r) and so it is equivalent to the normalized Lebesgue measure on B_i. Moreover within a fixed number of iterations it will regularize as explained

in Remark 3.13 and so the image of $(B_i, \tilde{\nu}_i)$ will be a standard family.
Consider the set $B_i \setminus \sigma$ that remains after the coupling step. We will treat each of these objects in the same way so from now on we omit the subscript i. Our general strategy is to show that as we iterate $B \setminus \sigma$, more and more parts of the image will become standard families, so we can apply our coupling machinery to them. We also would like to understand what proportion of the points became members of standard families up to time n. The basic ideas of the constructions in the rest of the paper date back to [6], however we need additional care.
Under the action of the dynamics $B \setminus \sigma$ will be expanded and cut by singularities and also artificially to keep the diameter of the components below δ_0 (see Remark 3.3). At certain moments some of the connected components of the image will be disjoint from the image of σ, so they will have nice boundaries and they will form a standard family. We will use the expression that these components are *released* at this moment of iteration. First we define a *release time* for points in $B \setminus \sigma$. This will be a function $f : B \setminus \sigma \to \mathbb{N}$ defined separately for two types of points.
Release time for type I points: these are points $x \in B \setminus \sigma$ such that the local stable manifold $W^s(x)$ intersects W^u_σ, the unstable manifold on which the special magnet σ was built. We will use the notation $h(x) = W^s(x) \cap W^u_\sigma$. Then $h(x) \notin W^1_{\sigma,\infty}$, where $W^1_{\sigma,\infty}$ is the limit object in the δ_2-filtration of $W^u_\sigma(\delta_1/3)$ (the central part of W^u_σ with radius $\delta_1/3$), thus $W^u_\sigma \cap \sigma = W^1_{\sigma,\infty}$. Hence, either $h(x) \in W^u_\sigma \setminus W^u_\sigma(\delta_1/3)$ or $h(x) \in W^0_{\sigma,m}$ for some $m = m(x) \geq 0$, where $W^0_{\sigma,m}$ is a gap in the δ_2-filtration of $W^u_\sigma(\delta_1/3)$. In the former case we set $m(x) = 0$ and $\varepsilon(x) = d_{W^u_\sigma}(h(x), W^u_\sigma(\delta_1/3))$. In the later case $m(x)$ is already determined and we set $\varepsilon(x) = d(F^m(h(x)), \partial F^m(W^0_{\sigma,m}))$. Then we define the release time to be $f(x) = m(x) + \log_\Lambda((\delta_0 + 4\delta_3)/\varepsilon(x))$.
Release time for type II points: these are points in $B \setminus \sigma$ whose local stable manifolds does not intersect W^u_σ. Recall that in the proof of Lemma 5.7 we showed that $W^u_\sigma(\delta_1/3)$ is overshadowed by B with stable distance at most $2\delta_3$.

!!!Inner Comment!!! 5.17. *Lemma 5.7 nem biztos, hogy létezni fog a változtatások után.*

Hence by our choice on the radius of W^u_σ (see the beginning of subsection 5.1), Lemma 3.7 implies that W^u_σ overshadows B with stable distance

$$d^s(B, W^u_\sigma) \leq 2(\frac{2}{C_t} + \frac{12\delta_3}{C_t\delta_1} + 1) \cdot 2\delta_3 \leq 4(\frac{2.1}{C_t} + 1) \cdot \delta_3 = C_0 \cdot \delta_3, \qquad (5.15)$$

where the second inequality follows from (3.4) and the final equality is just the definition of the constant C_0. From this it follows that if $x \in B \setminus \sigma$ is a point of type II, then its local stable manifold $W^s(x)$ does not contain a stable disk of radius $C_0\delta_3$. Therefore the quantity $m = m(x) = \min\{n \geq 0 : d(F^n(x), \partial \cup \xi^s_1) \leq C_0\delta_3\Lambda^{-n}\}$ is finite. Now consider $F^m(B)(F^m(x))$ the connected component of $F^m(B)$ that contains $F^m(x)$. We will show that on this component the $\delta_2\Lambda^{-m}/2$-neighborhood of $F^m(x)$ does not contain any point of the set $F^m(\sigma)$. Indeed, assume that, on the contrary, $\exists y \in F^m(B \cap \sigma)$ such that $d(y, F^m(x)) \leq \delta_2\Lambda^{-m}/2$. Then by construction the intersection $y' = W^s(y) \cap F^m(W^u_\sigma)$ exists, it belongs to $F^m(\sigma)$ and $d(y,y') \leq 2\delta_3 \cdot \Lambda^{-m}$. From this it follows that

$$\begin{aligned} d(y',\partial \cup \xi_1^s) &\leq d(y',y) + d(y,F^m(x)) + d(F^m(x),\partial \cup \xi_1^s) \leq \\ &\leq 2\delta_3 \cdot \Lambda^{-m} + \delta_2\Lambda^{-m}/2 + C_0\delta_3 \cdot \Lambda^{-m} = ((2+C_0)\delta_3 + \delta_2/2)\Lambda^{-m} \\ &\leq \delta_2 \cdot \Lambda^{-m} \end{aligned} \tag{5.16}$$

if we set the constant c_s in (3.4) to be $c_s = \frac{1}{2(2+C_0)}$. This is however a contradiction, because $y' \in F^m(W^u_\sigma \cap \sigma)$ and so by construction y' can not be that close to a singularity as (5.16) tells. We then define the release time to be $f(x) = 2m(x) + \log_\Lambda(2\delta_0/\delta_2)$.

Observe that $f(x)$ is defined in a way (for either type of points) that for any $n \geq f(x)$ either the connected component of $F^n(B)$ that contains $F^n(x)$ is disjoint from $F^n(\sigma)$ or the δ_0 neighborhood of $F^n(x)$ (measured along this component) does not contain any point of the set $F^n(\sigma)$. This implies that if we consider a 0-filtration $\{B_n^1\}$ of B, then the component of $F^n(B_n^1)$ that contains $F^n(x)$ does not intersect $F^n(\sigma)$. Thus this component, with the appropriately rescaled and restricted measure $F^n_*\tilde{\nu}$ on it will be a standard pair.

Now as an intermediate step of the construction of the coupling time we prove that the points $x \in B \setminus \sigma$ are released at an exponential rate.

Lemma 5.18. *There are constants $C_1 > 0$ and $\theta_1 \in (0,1)$ such that for each ball B after the coupling step we have*

$$\tilde{\nu}(x \in B \setminus \sigma | f(x) > n) \leq C_1\theta_1^n, \quad \forall n \geq 0.$$

Proof. First of all recall that the measure $\tilde{\nu}$ is uniformly equivalent to the normalized Lebesgue measure on B, so it is enough to prove the statement with m_B. We prove the bound separately for the two types of points. For each point x of type I we considered $h(x) = W^s(x) \cap W^u_\sigma$. By the absolute continuity of the holonomy map it suffices to estimate $m_{W^u_\sigma}(h(x)|f(x) > n)$. Recall that $f(x)$ was defined with the help of two numbers $m(x)$ and $\varepsilon(x)$. The argument can be decomposed into three parts:

1. If $h(x) \notin W^u_\sigma(\delta_1/3)$, then $m(x) = 0$ and so $f(x) > n$ is equivalent to $\varepsilon(x) < (\delta_0 + 4\delta_3)\Lambda^{-n}$. Therefore one has to estimate the measure of a d_u-dimensional spherical shell of width $(\delta_0 + 4\delta_3)\Lambda^{-n}$. Since unstable manifolds have bounded curvature it follows that

   $$m_{W^u_\sigma}(h(x) \notin W^u_\sigma(\delta_1/3) | d(h(x), W^u_\sigma(\delta_1/3)) < (\delta_0 + 4\delta_3)\Lambda^{-n}) \leq const \cdot \Lambda^{-n},$$

2. if $h(x) \in W^u_\sigma(\delta_1/3)$ then it belongs to the gap $W^0_{\sigma,m(x)}$ of the δ_2-filtration of $W^u_\sigma(\delta_1/3)$. First we consider the cases when $0 \leq m(x) \leq n/2$. We use the definition of the Z-function, part 1 of Corollary 3.6 and (3.1) in the

final line to conclude that

$$\begin{aligned}
&m_{W^u_\sigma}(h(x)\in W^u_\sigma(\delta_1/3)|f(x)>n, m(x)\le n/2) = \\
&= \sum_{m=0}^{n/2} m_{W^u_\sigma}(h(x)\in W^u_\sigma(\delta_1/3)|f(x)>n, m(x)=m) \le \\
&\le \sum_{m=0}^{n/2} Z[W^u_\sigma(\delta_1/3), W^0_{\sigma,m}, m]\cdot m_{W^u_\sigma}(W^u_\sigma(\delta_1/3))(\delta_0+4\delta_3)\Lambda^{-(n-m)} \le \\
&\le (3K_0+1)\bar{Z}_0(\delta_0+4\delta_3)\Lambda^{-n}\frac{\Lambda^{n/2+1}-1}{\Lambda-1} \le const\cdot\Lambda^{-n/2},
\end{aligned} \tag{5.17}$$

3. finally if $h(x)$ belongs to a gap with $m(x)>n/2$ then by part 3 of Corollary 3.6 and again by (3.1) we have

$$\begin{aligned}
&m_{W^u_\sigma}(h(x)|f(x)>n, m(x)>n/2) \le m_{W^u_\sigma}(h(x)|m(x)>n/2) = \\
&= \sum_{m=n/2+1}^{\infty} m_{W^u_\sigma}(W^0_{\sigma,m}) \le m_{W^u_\sigma}(W^u_\sigma(\delta_1/3))\cdot \sum_{m=n/2+1}^{\infty} C''\bar{Z}_0\delta_2\Lambda^{-m} \le \\
&\le const\cdot\Lambda^{-n/2}.
\end{aligned} \tag{5.18}$$

This completes the verification of the exponential bound with $\theta_1=\Lambda^{-1/2}$ for type I points.
For points of type II consider a $C_0\delta_3$-filtration $\{B^1_n, B^0_n\}$ of the u-manifold B. Observe that if x is a point of type II with $m(x)=m$ then $x\in B^0_m$. The u-manifold B is a ball (in its inner metric) with radius $\delta_1/3+2\delta_3$ which is slightly larger than $\delta_1/3$ so the analogue of (3.1) still holds, hence $\bar{Z}_0$ is still bounded. Then again use part 3 of Corollary 3.6 with the present parameters to conclude that $m_B(B^0_m)$ is exponentially small in m. □

Remember that our aim is to release connected components of the image of $B\backslash\sigma$, i.e. to collect connected components of the image that are disjoint from the image of σ. Consider a 0-filtration of B and denote it by $\{B_n\}$. Observe that we defined the point release time $f(x)$ in a way that if V is a connected component of B_n and $\exists x\in V$ such that $f(x)\le n$ then V is an admissable u-manifold and so with the image of the measure $\tilde{\nu}$ (restricted on it and normalized) it forms a standard pair. We say that V is released. Now we define a *component release time* $s: B\setminus\sigma\to\mathbb{N}$ in the following way. For a connected component $V\subset B_n$ we set $s(x)=n$ for every point $x\in V$ iff $\exists y\in F^n(V)$ such that $f(y)=n$ and $\forall k\in\{0,\dots,n-1\}$ the set $B_k(V)$, which is the connected component of B_k that contains V, has the property that $\forall y\in F^k(B_k(V))$ we have $f(y)>k$. It is obvious then that $s(x)\le f(x)$ for all $x\in B\setminus\sigma$.
For a fixed $s\in\mathbb{N}$ consider the set

$$\tilde{W}(s)=\cup\{V\subset F^s(B_s)|\forall x\in F^{-s}(V): s(x)=s\},$$

i.e. the union of the image of all components that are released exactly at time s. Then by construction of the release times $\tilde{W}(s)$ is a standard family. In order to guarantee that this family will return to the magnet and so we can proceed with the coupling of measures we want to use Corollary 5.9. This however requires the standard family to be proper. Since $\tilde{W}(s)$ does not necessarily satisfy this requirement we have to wait some time until it recovers and becomes a proper standard family. For a fixed s we define on $F^{-s}(\tilde{W}(s)) \subset B \setminus \sigma$ the function

$$g(x) = g(s) := \min\{n \geq 0 | \forall n' \geq n : F^{n'}(\tilde{W}(s)) \text{ is a proper standard family } \}.$$

It follows from Corollary 3.16 that $g(s) \leq \chi \cdot \ln Z[\tilde{W}(s), \tilde{W}(s), 0]$. We introduce the notation $q(s) = \tilde{\nu}(F^{-s}(\tilde{W}(s)))$ and discuss the relation of it with $g(s)$.

$$\begin{aligned} Z[\tilde{W}(s), \tilde{W}(s), 0] &= \sup_{\varepsilon>0} \frac{F^s_* \tilde{\nu}(x \in \tilde{W}(s) | d(x, \partial \tilde{W}(s)) < \varepsilon)}{\varepsilon \cdot F^s_* \tilde{\nu}(\tilde{W}(s))} \leq \\ &\leq \sup_{\varepsilon>0} \frac{F^s_* \tilde{\nu}(x \in F^s(B_s) | d(x, \partial F^s(B_s)) < \varepsilon)}{\varepsilon \cdot F^s_* \tilde{\nu}(F^s(B_s))} \cdot \frac{F^s_* \tilde{\nu}(F^s(B_s))}{F^s_* \tilde{\nu}(\tilde{W}(s))} = \\ &= \frac{Z^1_s}{\tilde{\nu}(F^{-s}(\tilde{W}(s)))} \leq \frac{\bar{Z}_0}{q(s)}, \end{aligned} \tag{5.19}$$

where in the final line we used part 1 of Corollary 3.6. Here $\bar{Z}_0$ is again bounded by the easy modification of (3.1). As a consequence, if the time we have to wait for the standard family, released at time s, to grow and become proper is more than n (i.e. if $g(s) > n$), then the relation

$$n \leq \chi \cdot \ln Z[\tilde{W}(s), \tilde{W}(s), 0] \leq \chi \ln \bar{Z}_0 - \chi \ln q(s)$$

holds. Hence

$$g(s) > n \quad \Rightarrow \quad q(s) < Ce^{-\frac{n}{\chi}}, \tag{5.20}$$

that is we have to wait a long time for the growth only on a set with exponentially small measure.

Finally we define the *recovery time* for every point $x \in B \setminus \sigma$ as $s(x) + g(x)$. Then for any fixed n the set $\{x \in B \setminus \sigma | s(x) + g(x) = n\}$ is either empty or forms a proper standard family with the probability measure induced by $\tilde{\nu}$ on it. As the key step of the coupling time construction we prove that the recovery time admits an exponential tail bound.

Lemma 5.19. *There are constants $C_2 > 0$ and $\theta_2 \in (0, 1)$ such that for every $n \geq 0$ we have*

$$\tilde{\nu}(x \in B \setminus \sigma | s(x) + g(x) > n) \leq C_2 \theta_2^n.$$

Proof. We use Lemma 5.18 and the implication (5.20) to deduce that

$$\begin{aligned} \tilde{\nu}(x | s(x) + g(x) > n) &\leq \tilde{\nu}(x | s(x) \geq n/2) + \tilde{\nu}(x | s(x) < n/2, g(x) > n/2) \leq \\ &\leq C_1 \theta_1^{n/2} + const \cdot \frac{n}{2} \cdot e^{-\frac{n}{2\chi}} \leq const \cdot \theta_2^n, \end{aligned}$$

where we choose θ_2 such that $\max\{\sqrt{\theta_1}, e^{-\frac{1}{2\chi}}\} < \theta_2 < 1$ holds. □

The rest of the construction of the coupling time goes in the exact same way as in [8, Appendix, p.1090]. Therefore we do not repeat it here in all detail, we just sketch the main lines of it instead. Before that we give a brief summary of

what we have done so far.
Recall that we had two proper standard families $\mathcal{G} = \{(W_i, \nu_i)\}_{i\in I}$ and $\mathcal{E} = \{(W_j, \nu_j)\}_{j\in J}$. After n_0 iterations some of the components of $F^{n_0}(W_k)$'s (k can be i or j) crossed the magnet and the preimage of the intersections formed subsets of the original families with $\mu_{\mathcal{G}}$-measure (or $\mu_{\mathcal{E}}$ respectively) at least d_0. We coupled exactly $d_0/2$ amount of the measures. From now on we will only talk about the image of the first family, analogous statements hold for the second one too. After the coupling we were left with three types of objects

1. standard pairs coming from components that had not been affected by the coupling,

2. admissible u-manifolds with dynamically Hölder continuous densities supported on them, but with regularity constant larger than C_r. They are almost standard pairs and within a fixed number of iterations the densities regularize and they eventually become standard families,

3. u-manifolds which arose by removing Cantor sets from balls and on these we have similar densities as in the previous item.

Let us denote the union of all these kind of objects we have after the first coupling, together with the induced measures on them by $\mathcal{G}_1$. We emphasize that $\mathcal{G}_1$ is *not* a standard family. The original measure $\mu_{\mathcal{G}}$ induces a measure on $\mathcal{G}_1$ (by this we mean not only the probability measures on the connected u-manifolds, but also the probability factor measure on the set of them). This will not be a probability measure since we removed $d_0/2$ amount of the mass by coupling. So we normalize this induced measure and denote the resulting probability measure by $\mu_{\mathcal{G}_1}$. As we said $\mathcal{G}_1$ is not a standard family. One key part of what we wrote so far in the present subsection was to show that under the action of F more and more parts of $\mathcal{G}_1$ will become standard, moreover *proper* standard families. For almost every point x that belongs to a u-manifold of a pair from $\mathcal{G}_1$ we introduced a recovery time. This is the time when the u-manifold that contains x, together with the probability measure on it induced by $\mu_{\mathcal{G}_1}$, is a standard pair, moreover it can be regarded as a member of a *proper* standard family. This recovery time is actually a fixed constant for points that belong to type 1 or type 2 kind of objects (cf. (5.14)) and it was defined to be $s(x) + g(x)$ for points that belong to type 3 objects. Corollary 5.9 implies that whenever a point is recovered, for all $n \geq n_0$ the n-th image of the proper standard family to which it belongs to will cross the magnet σ and the intersection will have at least d_0 relative measure within the family.

!!!Inner Comment!!! 5.20. *Az előző mondat megint nem lesz igaz, csak az, hogy n_0-as blokkonként lesz egy-egy pillanat, amikor legalább d_0 mérték van a mágnesen.*

We will use the expression that in such a situation this standard family can be *stopped* and $d_0/2$ amount of its mass can be coupled. From our estimates in the present subsection it follows that

$$\mu_{\mathcal{G}_1}(x| \text{ recovery time of } x > n) \leq const \cdot \theta_2^n. \tag{5.21}$$

Of course the actual return time for points of two different initial standard families may be very different. This is another point where we use the cylindrical

extensions of the objects: this way we can define a stopping time on which we have uniform control. The stopping time of a point (x, t) must be at least the recovery time of x plus n_0, because in this way (as we explained before) we guarantee that (x, t) belongs to a proper standard family crossing the magnet in a set with fix relative measure. We do not have to be gready though! We may decide not to stop a point when this situation happens for the first time, we may iterate it forward instead and stop it later at an appropriate moment. This means that we have some freedom in the way how the stopping time is defined. The content of Lemma 5.22 below is that the stopping time can be defined in such a way that it has the same distribution for the cylindrical extension of objects from any $\mathcal{G}_1$.
Let $\hat{\mu}_{\mathcal{G}_1}$ denote the product measure on $\mathcal{G}_1 \times [0, 1]$ which is just $\mu_{\mathcal{G}_1}$ times the uniform measure on $[0, 1]$.

!!!Inner Comment!!! 5.21. *Itt korrigálni kell aszerint, hogy q_n ne az n-ik pillanatban megállított pontok mértéke legyen, hanem az n-ik n_0-as blokkban megállítottaké. Emiatt lehet, hogy a jelenleginél egy kicsit részletesebb magyarázatra is szükség lesz.*

Lemma 5.22. *There exists a probability distribution q_n, $n = 1, \dots,$ independent of $\mathcal{G}_1$ that satisfies $q_n < const \cdot \theta_2^n$ and for all $n \in \mathbb{N}$ we have*

$$\hat{\mu}_{\mathcal{G}_1}((x, t)| \text{ the stopping time of } (x, t) = n) = q_n.$$

The proof of this lemma can be found in [8, Appendix] after Proposition A.5., nonetheless here we provide a short explanation. The tail bound on q_n follows from (5.21), while the fact that q_n can be chosen independently of $\mathcal{G}_1$ is based on the large amount of freedom we have at the definition of the stopping process. Indeed, (5.21) guarantees that at certain time moments we have some amount of measure that can be stopped regardless of what $\mathcal{G}_1$ is. If this amount is more than what we want to stop, then we just cut some of our cylinders in proper standard families horizontally to ensure that we stop exactly the amount of mass we need.
Note that for any n the collection of points that are stopped at time n forms a union of proper standard families. Each of these families crosses the magnet and the relative measure of the intersections with the magnet is at least d_0. Hence, for every n, we can apply our coupling procedure and couple $d_0/2$ amount of the points that are stopped at time n and for these points we set the coupling time $\Gamma(x, t)$ to be the initial n_0 plus the stopping time of (x, t). Then we inductively repeat the construction of recovery and stopping times on the set that remains after this second coupling. Note that the points that are stopped once are not necessarily coupled at that moment and so they may be stopped several more times before they finally get coupled. The coupling time $\Gamma(x, t)$ of the point (x, t) will be n_0 plus the sum of all stopping times defined for it before it got coupled.
To finish our argument we recall that $\mathcal{G} = \{(W_i, \nu_i)\}_{i \in I}$ was one of our initial proper standard families. As in [8, Appendix] we introduce the following notations. Let $\bar{p}_n := \hat{\mu}_{\mathcal{G}}((x, t) \in \cup_{i \in I} \hat{W}_i | \Gamma(x, t) = n)$, the fraction of points being coupled exactly at time n and let $p_n := \hat{\mu}_{\mathcal{G}}((x, t) \in \cup_{i \in I} \hat{W}_i | (x, t)$ is stopped at time $n)$, the fraction of points stopped (not necessarily for the first time) at time n. Note that $\sum_{n=1}^{\infty} \bar{p}_n = 1$, however $\sum_{n=1}^{\infty} p_n$ may be larger because of multiple stoppings, in

fact $\bar{p}_n = d_0/2 \cdot p_n$. Furthermore, the above described construction implies the convolution law:

$$p_{n+n_0} = (1-d_0/2)\left(q_n + (1-d_0/2)\sum_{i=1}^{n-1} q_{n-i}p_{n_0+i}\right) \qquad \forall n \geq 1. \tag{5.22}$$

Using this and the tail bound on the sequence q_n a standard argument (as in the lines after [8, (A.27)]) using generator functions gives that $\bar{p}_n$ has an exponentially decreasing tail, which verifies the second part of the coupling lemma (Lemma 5.2).

6 Acknowledgement

Both authors were supported by the OTKA grant K 104745.

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