Differential equations, Sample test 2

Topics: Higher order linear nonhomogeneous equations, Laplace transform, power series method, linear systems.

1. (5 points)

Find the solution of the differential equation $y''(x) - 3y'(x) - 4y(x) = 16x + e^{-x}$.

2. (4 points)

$y'(x) = (x^4 - 1)y(x) + x y^2(x), \quad y(0) = 2$. Find the third degree Taylor polynomial of the solution to the differential equation.

3. (4 points)

Solve the following initial value problem by the Laplace transform.

$y''(t) - 8y'(t) + 7y(t) = 0, \; y(0) = 1, \; y'(0) = -4$

4. (5+5 points)

Solve the following initial value problems using the eigenvectors and eigenvalues of the coefficient matrix.
Optional for 4 points: solve exercise b) by the Laplace transform.

a) $x'(t) = 4x(t) + 3y(t), \; y'(t) = x(t) + 2y(t), \; x(0) = 4, \; y(0) = 2$

b) $x'(t) = 5x(t) - 2y(t), \; y'(t) = x(t) + 3y(t), \; x(0) = 1, \; y(0) = -2$

Differential equations, Sample test 2, solutions

1. (5 points)

Find the solution of the differential equation $y''(x) - 3y'(x) - 4y(x) = 16x + e^{-x}$.

Solution: $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \implies \lambda_1 = 4,\ \lambda_2 = -1$

The general solution of the homogeneous equation is: $y_h(x) = c_1 e^{4x} + c_2 e^{-x}$

The particular solution of the nonhomogeneous equation:

$$
\begin{aligned}
-4\cdot &\mid y_p(x) = (Ax + B) + C x e^{-x} \Longleftarrow \text{outer resonance}\\
-3\cdot &\mid y_p'(x) = A + C e^{-x} - C x e^{-x}\\
1\cdot &\mid y_p''(x) = -C e^{-x} - Ce^{-x} + C x e^{-x}
\end{aligned}
$$

Substituting into the nonhomogeneous equation:

$$x e^{-x}(-4C + 3C + C) + e^{-x}(-3C - 2C) + x\cdot(-4A) + (-4B - 3A) = 16x + e^{-x}$$

$$
\begin{aligned}
&-3C - 2C = 1 && \implies C = -\frac{1}{5}\\
&-4A = 16 && A = -4\\
&-4B - 3A = 0 && B = 3
\end{aligned}
$$

The general solution of the nonhomogeneous equation is:

$$y(x) = y_h(x) + y_p(x) = c_1 e^{4x} + c_2 e^{-x} + (-4x + 3) - \frac{1}{5} x e^{-x}$$

2. (4 points)

$y'(x) = (x^4 - 1)\,y(x) + x\,y^2(x),\quad y(0) = 2$. Find the third degree Taylor polynomial of the solution to the differential equation.

Solution:

$$\implies y'(0) = (0^4 - 1)\,y(0) + 0\cdot y^2(0) = -2$$

$$y''(x) = 4x^3 y(x) + (x^4 - 1)\,y'(x) + y^2(x) + x\cdot 2y(x)\,y'(x)$$

$$\implies y''(0) = 4\cdot 0\cdot 2 + (0 - 1)(-2) + 2^2 + 0\cdot 2\cdot 2\cdot(-2) = 6$$

$$y'''(x) = 12x^2 y(x) + 4x^3 y'(x) + 4x^3 y'(x) + (x^4 - 1)\,y''(x) + 2y(x)\,y'(x)$$
$$+2y(x)\,y'(x) + 2x\,y'(x)\,y'(x) + 2x\,y(x)\,y''(x)$$

$$\implies y'''(x) = 0 + 0 + 0 + (0 - 1)\cdot 6 + 2\cdot 2\cdot(-2) + 2\cdot 2\cdot(-2) + 0 + 0 = -22$$

$$y(x) \approx T_3(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}\,x^2 + \frac{y'''(0)}{3!}\,x^3 =$$

$$= 2 - 2x + \frac{6}{2!}\,x^2 + \frac{-22}{3!}\,x^3 = 2 - 2x + 3x^2 - \frac{11}{3}\,x^3$$

3. (4 points)

Solve the following initial value problem by the Laplace transform.

$y''(t) - 8\,y'(t) + 7\,y(t) = 0,\ \ y(0) = 1,\ y'(0) = -4$

Solution:

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \implies \mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0),\ \mathcal{L}\{y''(t)\}(s) = s^2\,Y(s) - s\,y(0) - y'(0)$

The Laplace transform of the differential equation is

$$\left(s^2\,Y(s) - s\cdot 1 - (-4)\right) - 8\,(s\,Y(s) - 1) + 7\,Y(s) = 0$$
$$Y(s)\left(s^2 - 8s + 7\right) - s + 4 + 8 = 0$$

$$Y(s) = \frac{s-12}{s^2 - 8s + 7} = \frac{s-12}{(s-1)(s-7)} = \frac{A}{s-1} + \frac{B}{s-7}$$

$\implies s - 12 = A(s-7) + B(s-1)$

if $s = 7$ then $-5 = 0 + B\cdot 6 \implies B = -\frac{5}{6}$

if $s = 1$ then $-11 = A\cdot(-6) + 0 \implies A = \frac{11}{6}$

$$\implies Y(s) = \frac{11}{6}\cdot\frac{1}{s-1} - \frac{5}{6}\cdot\frac{1}{s-7}$$

The solution of the initial value problem is the inverse Laplace transform of $Y(s)$:

$$y(t) = \frac{11}{6}\,e^t - \frac{5}{6}\,e^{7t}$$

4. (5+5 points)

Solve the following initial value problems using the eigenvectors and eigenvalues of the coefficient matrix.

Optional for 4 points: solve exercise b) by the Laplace transform.

a) $x'(t) = 4\,x(t) + 3\,y(t),\ \ y'(t) = x(t) + 2\,y(t),\ \ x(0) = 4,\ \ y(0) = 2$

b) $x'(t) = 5\,x(t) - 2\,y(t),\ \ y'(t) = x(t) + 3\,y(t),\ \ x(0) = 1,\ \ y(0) = -2$

Solution. a)

$x_1' = 4x_1 + 3x_2 \quad x_1(0) = 4 \implies$ the coefficient matrix is $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$

$x_2' = x_1 + 2x_2 \quad x_2(0) = 2$

The eigenvalues of A:

$\det(A - \lambda I_2) = \det\begin{pmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{pmatrix} = (4-\lambda)(2-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5) = 0 \implies \lambda_1 = 1,\ \lambda_2 = 5$

The eigenvectors of A:

Case 1) If $\lambda_1 = 1$ and $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ then $A\boldsymbol{u} = \lambda_1 \boldsymbol{u} \iff \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 1 \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \iff$

$4u_1 + 3u_2 = u_1 \iff u_1 = -u_2 \implies u_2 = 1,\ u_1 = -1 \implies \boldsymbol{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$u_1 + 2u_2 = u_2$

Case 2) If $\lambda_2 = 5$ and $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $A\boldsymbol{v} = \lambda_2 \boldsymbol{v} \iff \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 5 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff$

$4v_1 + 3v_2 = 5v_1 \iff v_1 = 3v_2 \implies v_2 = 1,\ v_1 = 3 \implies \boldsymbol{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$v_1 + 2v_2 = 5v_2$

The general solution of the differential equation system is:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = P e^{Bt} \boldsymbol{c} =$$

$$== \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^t \\ c_2 e^{5t} \end{pmatrix} = \begin{pmatrix} -c_1 e^t + 3c_2 e^{5t} \\ c_1 e^t + c_2 e^{5t} \end{pmatrix}$$

or:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \boldsymbol{u} + c_2 e^{\lambda_2 t} \boldsymbol{v} =$$

$$= c_1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} e^{\lambda_1 t} + c_2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda_2 t} = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t} = \begin{pmatrix} -c_1 e^t + 3c_2 e^{5t} \\ c_1 e^t + c_2 e^{5t} \end{pmatrix}$$

The general solution: $x_1(t) = -c_1 e^t + 3c_2 e^{5t}$

$x_2(t) = c_1 e^t + c_2 e^{5t}$

From the initial conditions: $x_1(0) = 4 \implies -c_1 + 3c_2 = 4 \implies c_1 = \frac{1}{2},\ c_2 = \frac{3}{2}$

$x_2(0) = 2 \qquad c_1 + c_2 = 2$

The solution of the initial value problem: $x_1(t) = -\frac{1}{2} e^t + \frac{9}{2} e^{5t}$

$x_2(t) = \frac{1}{2} e^t + \frac{3}{2} e^{5t}$

b) $x_1' = 5x_1 - 2x_2 \quad x_1(0) = 1 \quad \Longrightarrow$ the coefficient matrix is $A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$

$x_2' = x_1 + 3x_2 \qquad x_2(0) = -2$

The eigenvalues of A:

$$\det(A - \lambda I_2) = \det\begin{pmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix} = (5-\lambda)(3-\lambda) + 2 = \lambda^2 - 8\lambda + 17 = 0 \Longrightarrow \lambda_{1,2} = 4 \pm i$$

Let $\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be the complex eigenvector corresponding to $\lambda_1 = 4 + i$. Then

$$A\,\boldsymbol{p} = \lambda_1 \boldsymbol{p} \iff \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (4+i)\cdot\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \iff \begin{matrix} 5p_1 - 2p_2 = (4+i)p_1 \\ p_1 + 3p_2 = (4+i)p_2 \end{matrix} \iff \begin{matrix} p_1 - 2p_2 = i\,p_1 \\ p_1 = p_2 + i\,p_2 \end{matrix} \iff$$

$p_1(1-i) = 2p_2$

$p_1 = (1+i)p_2$

$$\Longrightarrow p_1 = 1+i,\; p_2 = 1 \Longrightarrow \boldsymbol{p} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1\cdot i \\ 1+0\cdot i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\Longrightarrow$ The real and imaginary parts of $\boldsymbol{p}$ are: $\boldsymbol{u} = \mathrm{Re}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v} = \mathrm{Im}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The general solution of the differential equation system is $(\lambda_{1,2} = 4 \pm i = \alpha \pm \beta i)$:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = P\,e^{Bt}\,\boldsymbol{c} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} e^{4t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$$

$$e^{4t}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1\cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix} = e^{4t}\begin{pmatrix} (c_1\cos t + c_2\sin t) + (-c_1\sin t + c_2\cos t) \\ c_1\cos t + c_2\sin t \end{pmatrix}$$

The general solution:

$\Longrightarrow x_1(t) = (c_1 + c_2)e^{4t}\cos t + (-c_1 + c_2)e^{4t}\sin t$

$x_2(t) = c_1 e^{4t}\cos t + c_2 e^{4t}\sin t$

or:

The complex solution is (using that $e^{it} = \cos t + i\sin t$):

$$\boldsymbol{x}_{\text{complex}}(t) = e^{\lambda_1 t}\,\boldsymbol{p} = e^{(4+i)t}\begin{pmatrix} 1+i \\ 1 \end{pmatrix} = e^{4t}(\cos t + i\sin t)\begin{pmatrix} 1+i \\ 1 \end{pmatrix} =$$

$$= e^{4t}\begin{pmatrix} (\cos t - \sin t) + i(\cos t + \sin t) \\ \cos t + i\sin t \end{pmatrix} = e^{4t}\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + i\cdot e^{4t}\begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}$$

It can be shown that $\mathrm{Re}(\boldsymbol{x}_{\text{complex}}(t)) = e^{4t}\begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix}$ and $\mathrm{Im}(\boldsymbol{x}_{\text{complex}}(t)) = e^{4t}\begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}$

are also solutions and linearly independent. Since the dimension of the linear space generated by the solutions has dimension 2 then these two functions constitute a basis, so the general solution of the differential equation system is:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1\, e^{4t} \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2\, e^{4t} \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix} \Longrightarrow$$

$x_1(t) = c_1\, e^{4t} (\cos t - \sin t) + c_2\, e^{4t} (\cos t + \sin t) = (c_1 + c_2)\, e^{4t} \cos t + (-c_1 + c_2)\, e^{4t} \sin t$
$x_2(t) = c_1\, e^{4t} \cos t + c_2\, e^{4t} \sin t$

From the initial conditions: $x_1(0) = 1 \quad \Longrightarrow \quad (c_1 + c_2) + 0 = 1 \Longrightarrow c_1 = -2,\ c_2 = 3$
$x_2(0) = -2 \qquad c_1 + 0 = -2$

The solution of the initial value problem: $x_1(t) = e^{4t} \cos t + 5\, e^{4t} \sin t$
$x_2(t) = -2\, e^{4t} \cos t + 3\, e^{4t} \sin t$