Differential equations - Sample Test 1

Topics of the test are: separable differential equations, first order linear differential equations, second order linear homogeneous differential equations.

1. (5 poins)

Consider the equation $x'(t) = t^3(x(t) - 2)$.
a) What type of equation is this? Please choose exactly one answer from the following:
autonomous / second order / separable / third order / none of these
b) How many constant solutions are there?
c) Find the general solution in implicit and in explicit form.

2. (8 points)

Suppose that the amount of salt in a tank is described by the equation $y'(t) = 0.05\,(20 - y(t))$.
(The quantities are measured in SI units.)
a) Find the general solution in an explicit form.
If at time $t = 0$ the amount of salt is 0 kg then
b) what will be the solution $y(t)$?
c) in how many minutes will the tank contain 10 kg of salt?

3. (7 points)

Suppose that the amount of salt in the previous exercise is linearly increased and thus the equation is $y'(t) = (1 + b\,t) - 0.05\,y(t)$ where $b > 0$ is a constant. Find the general solution.

4. (8 points)

In an RLC circuit the current $I(t)$ as a function of time is described be the equation
$L\,I''(t) + R\,I'(t) + \frac{1}{C}\,I(t) = 0$. (The quantities are measured in SI units.) Let $L = 1$, $R = 5$, $C = 0.25$.
a) What are the roots of the characteristic equation?
b) What are the linearly independent solutions?
c) What is the general solution $I(t)$?
d) What is the solution if the initial conditions are $I(0) = 3$ and $I'(0) = 0$?

Differential equations - Sample Test 1, solutions

1. (5 poins)

Consider the equation $x'(t) = t^3(x(t) - 2)$.
a) What type of equation is this? Please choose exactly one answer from the following:
autonomous / second order / separable / third order / none of these
b) How many constant solutions are there?
c) Find the general solution in implicit and in explicit form.

Solution.
a) separable
b) There is one constant solution, $x(t) \equiv 2$.
c) The equation can be written as $\frac{dx}{dt} = t^3(x - 2)$
When $x(t) = \mp 2$ then by separating the variables: $\int \frac{1}{x-2}\,dx = \int t^3\,dt$
The general solution in an implicit form: $\ln|x - 2| = \frac{t^4}{4} + c$
$\Longrightarrow |x - 2| = e^{\frac{t^4}{4} + c} = e^c \cdot e^{\frac{t^4}{4}} \Longrightarrow x - 2 = \pm e^c \cdot e^{\frac{t^4}{4}} \Longrightarrow x = 2 \pm e^c \cdot e^{\frac{t^4}{4}}$

The general solution in an explicit form: $x(t) = 2 + C\,e^{\frac{t^4}{4}},\ C \in \mathbb{R}$.

2. (8 points)

Suppose that the amount of salt in a tank is described by the equation $y'(t) = 0.05\,(20 - y(t))$.
(The quantities are measured in SI units.)
a) Find the general solution in an explicit form.
If at time $t = 0$ the amount of salt is 0 kg then
b) what will be the solution $y(t)$?
c) in how many minutes will the tank contain 10 kg of salt?

Solution.
a) $\frac{dy}{dt} = 0.05\,(20 - y)$
Constant solution: $y \equiv 20$. If $y \neq 20$: $\int \frac{1}{20 - y}\,dy = \int 0.05\,dt$
The general (non-constant) solution in an implicit form: $-\ln|20 - y| = 0.05\,t + c_1$

$\Longrightarrow \ln|20 - y| = -0.05\,t - c_1$
$\Longrightarrow |20 - y| = e^{-0.05\,t - c_1} = e^{-c_1} \cdot e^{-0.05\,t} \Longrightarrow 20 - y = \pm e^{-c_1} \cdot e^{-0.05\,t}$
$\Longrightarrow y = 20 \pm e^{-c_1} \cdot e^{-0.05\,t}$ or $y \equiv 20$

The general solution in an explicit form: $y(t) = 20 + C\,e^{-0.05\,t},\;\; C \in \mathbb{R}$

b) $y(0) = 0 \implies 20 + C = 0 \implies C = -20$
The solution of the initial value problem is $y(t) = 20 - 20\,e^{-0.05\,t}$

c) $20 - 20\,e^{-0.05\,t} = 10 \implies 10 = 20\,e^{-0.05\,t} \implies e^{-0.05\,t} = 0.5 \implies -0.05\,t = \ln 0.5 = -\ln 2$
$\implies t = \dfrac{\ln 2}{0.05} = 20 \ln 2$ (minutes)

3. (7 points)

Suppose that the amount of salt in the previous exercise is linearly increased and thus the equation is $y'(t) = (1 + b\,t) - 0.05\,y(t)$ where $b > 0$ is a constant. Find the general solution.

Solution.

1st step: The homogeneous equation is $\dfrac{dy}{dt} = -0.05\,y$

Constant solution: $y \equiv 0$. If $y \neq 0$: $\int \frac{1}{y}\,dy = \int -0.05\,dt$

$\implies \ln|y| = -0.05\,t + c_1 \implies |y| = e^{-0.05\,t + c_1} = e^{c_1} \cdot e^{-0.05\,t}$

$\implies y = \pm e^{c_1} \cdot e^{-0.05\,t}$ or $y \equiv 0$

The general solution of the homogeneous equation is
$y_h(t) = C\,e^{-0.05\,t},\;\; C \in \mathbb{R}$.

2nd step: Applying the variation of the constant method, we assume that the nonhomogeneous equation has a particular solution of the form

$y_p(t) = c(t) \cdot e^{-0.05\,t}$

$\implies y_p'(t) = c'(t) \cdot e^{-0.05\,t} + c(t) \cdot e^{-0.05\,t} \cdot (-0.05)$

Substituting into the original equation:

$c'(t) \cdot e^{-0.05\,t} + c(t) \cdot e^{-0.05\,t} \cdot (-0.05) + 0.05 \cdot c(t) \cdot e^{-0.05\,t} = 1 + b\,t$

$c'(t) \cdot e^{-0.05\,t} = 1 + b\,t$

$c'(t) = (1 + b\,t)\,e^{0.05\,t}$

$$c(t) = \int (1 + b\,t)\,e^{0.05\,t}\,dt = \frac{e^{0.05\,t}}{0.05} \cdot (1 + b\,t) - \int \frac{e^{0.05\,t}}{0.05} \cdot b\,dt =$$
$$= \frac{e^{0.05\,t}}{0.05} \cdot (1 + b\,t) - \frac{e^{0.05\,t}}{0.05^2} \cdot b$$

Integration by parts: $f'(t) = e^{0.05\,t} \implies f(t) = \dfrac{e^{0.05\,t}}{0.05}$

$g(t) = 1 + b\,t \implies g'(t) = b$

The particular solution:

$$y_p(t) = c(t) \cdot e^{-0.05\,t} = \left(\frac{e^{0.05\,t}}{0.05} \cdot (1 + b\,t) - \frac{e^{0.05\,t}}{0.05^2} \cdot b \right) e^{-0.05\,t} = \frac{1}{0.05} \cdot (1 + b\,t) - \frac{1}{0.05^2} \cdot b =$$
$$= 20\,(1 + b\,t) - 400\,b$$

The general solution of the nonhomogeneous equation:

$$y(t) = y_h(t) + y_p(t) = C\,e^{-0.05\,t} + 20\,(1 + b\,t) - 400\,b, \;\; C \in \mathbb{R}.$$

4. (8 points)

In an RLC circuit the current $I(t)$ as a function of time is described be the equation $L\,I''(t) + R\,I'(t) + \frac{1}{C}\,I(t) = 0$. (The quantities are measured in SI units.) Let $L = 1$, $R = 5$, $C = 0.25$.

a) What are the roots of the characteristic equation?

b) What are the linearly independent solutions?

c) What is the general solution $I(t)$?

d) What is the solution if the initial conditions are $I(0) = 3$ and $I'(0) = 0$?

Solution. The equation is $I''(t) + 5\,I'(t) + 4\,I(t) = 0$.

a) The characteristic equation is $\lambda^2 + 5\,\lambda + 4 = (\lambda + 1)\,(\lambda + 4) = 0$
The roots of the characteristic equation are $\lambda_1 = -1$, $\lambda_2 = -4$

b) The linearly independent solutions are e^{-t} and $e^{-4\,t}$

c) The general solution is $I(t) = c_1\,e^{-t} + c_2\,e^{-4\,t}$

d) $I'(t) = -c_1\,e^{-t} - 4\,c_2\,e^{-4\,t}$

$I(0) = 3 \implies c_1 + c_2 = 3 \implies c_1 = 4,\; c_2 = -1$

$I'(0) = 0 \implies -c_1 - 4\,c_2 = 0$

The solution of the initial value problem is $I(t) = 4\,e^{-t} - e^{-4\,t}$