# 11 - Lyapunov’s method, first integrals

## Introduction

**Example.** Consider the following nonlinear system

$$x' = -y - x^3 = f_1(x, y)$$
$$y' = x - y^3 \quad = f_2(x, y)$$

The equilibrium point/points of the system can be determined by solving the following algebraic equation system:

$$\begin{array}{ll} x' = 0 & \Longrightarrow -y - x^3 = 0 \Longrightarrow -y - y^9 = -y\left(1 + y^8\right) = 0 \Longrightarrow x = 0,\ y = 0. \\ y' = 0 & \quad x - y^3 = 0 \end{array}$$

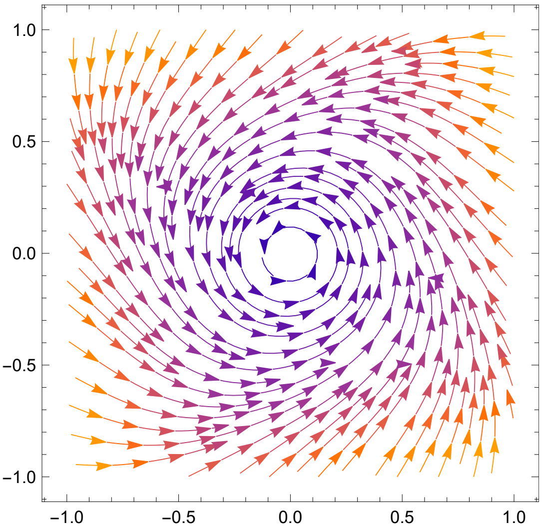
The only equilibrium point is the origin: (0, 0).

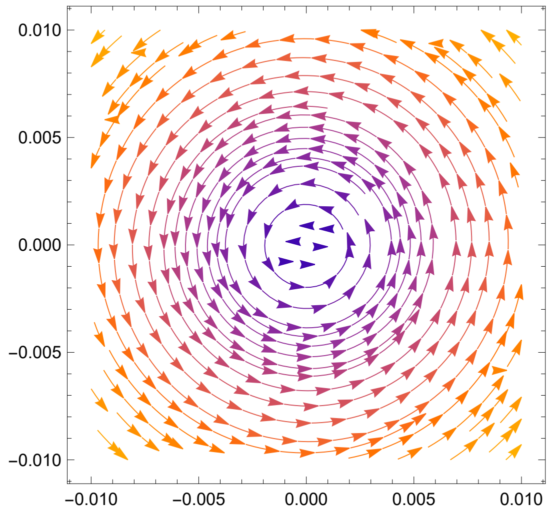
The Jacobian matrix is $f'(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{pmatrix}$

The coefficient matrix of the linearized system is $f'(0, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\Longrightarrow \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0 \Longrightarrow$ the eigenvalues are $\lambda_{12} = \pm i$.

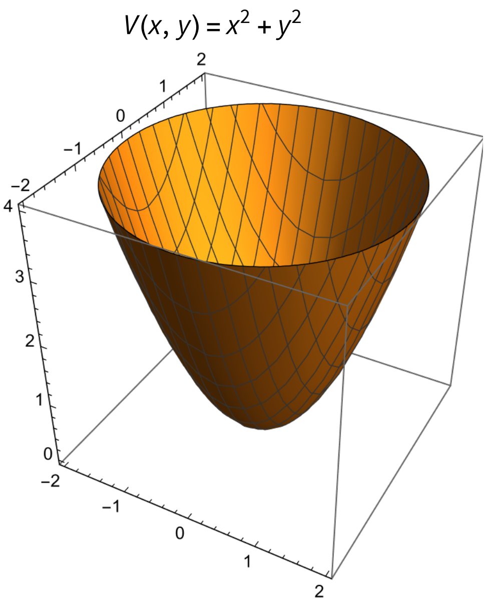
Since $\lambda_1$ and $\lambda_2$ are not hyperbolic (that is, $\text{Re}(\lambda_{12}) = 0$), the stability of the origin cannot be decided based on the eigenvalues. The phase portrait of the system is the following:

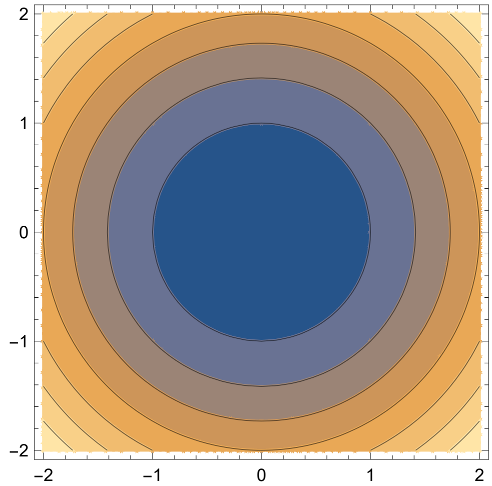



The trajectories go around the origin but it cannot be seen if they converge to the origin or diverge from it. To decide this question we will use Lyapunov’s method.

**Lyapunov's method:** We consider the function $V(x, y) = x^2 + y^2$, where $(x, y) \in \mathbb{R}^2$ and investigate whether the value of $V$ increases or decreases along the trajectories. The nullclines of $V$ are circles centered at the origin.



**Observation:** If the value of $V \begin{cases} \text{increases} \\ \text{decreases} \end{cases}$ along the trajectories then the trajectories $\begin{cases} \text{diverge from} \\ \text{converge to} \end{cases}$ the origin, so the origin is $\begin{cases} \text{unstable} \\ \text{asymptotically stable} \end{cases}$.

Let $(x(t),\ y(t)),\ (t \in \mathbb{R})$ be an arbitrary solution and let $V^*(t) = V(x(t), y(t)),\ (t \in \mathbb{R})$.

$V(x, y) = x^2 + y^2 \quad \left(\Longrightarrow\ \partial_x V(x, y) = 2x,\ \partial_y V(x, y) = 2y\right)$

Then the derivative of $V^*(t) = x^2(t) + y^2(t)$ with respect to the time $t$ is the following:

$$\frac{d}{dt} V^*(t) = 2x(t) \cdot x'(t) + 2y(t) \cdot y'(t) = 2x \cdot x' + 2y \cdot y' =$$
$$= \partial_x V(x, y) \cdot f_1(x, y) + \partial_y V(x, y) \cdot f_2(x, y) =$$
$$= 2x\left(-y - x^3\right) + 2y\left(x - y^3\right) = -2\left(x^4 + y^4\right) < 0 \text{ for all } (x, y) \neq (0, 0)$$

Since the derivative of $V$ is negative, then the value of $V$ decreases along the trajectories, so the trajectories intersect the nullclines of $V$ going inwards. It means that the trajectories converge towards the origin and thus the origin is asymptotically stable.
We obtained the phase portrait not only in a neighbourhood of the origin but on the whole phase plane.

This idea can be formulated for two-dimensional systems as follows.

## Lyapunov’s method

**Definition.** Let $M \subset \mathbb{R}^2$ be a region and let $f : M \to \mathbb{R}^2$ and $V : M \to \mathbb{R}$ be continuously differentiable functions. Then the derivative of $V$ with respect to the system

$$x' = f_1(x, y), \;\; y' = f_2(x, y)$$

(or the Lie-derivative of $V$) is the following scalar product:

$$L_f V(x, y) = V'(x, y) \cdot f(x, y) = \partial_x V(x, y) \cdot f_1(x, y) + \partial_y V(x, y) \cdot f_2(x, y)$$

where $f(x, y) = (f_1(x, y), f_2(x, y))$.

**Remark.** As we have seen in the example, if $(x(t), \; y(t))$ is a solution of the system $x' = f_1(x, y), \;\; y' = f_2(x, y)$ then for the function $V^*(t) = V(x(t), y(t))$, $\frac{d}{\mathrm{dt}} V^*(t) = L_f V(x(t), y(t))$.
It means that the sign of the function $L_f V(x, y)$ indicates whether the value of $V$ increases or decreases along the trajectories.
The main point of Lyapunov's method is to find a function $V$ that is monotonic along the trajectories and the monotonicity of $V$ can be decided without calculating the solutions.

In the following, we investigate the stability of the equilibrium point with the help of a Lyapunov function. Let $M \subset \mathbb{R}^2$ be a region, let $f : M \to \mathbb{R}^2$ be a continuously differentiable function and let $p \in M$ be an equilibrium point of the autonomous system $x'(t) = f(x(t))$, that is, $f(p) = 0$.

**Theorem (Lyapunov's stability theorem).** Assume that there exists an open neighbourhood $U \subset M$ of $p$ and a continuously differentiable function $V : U \to \mathbb{R}$ such that

1. $V(p) < V(q)$ for all $q \in U \setminus \{p\}$ and
2. $L_f V(q) \leq 0$ for all $q \in U \setminus \{p\}$.

Then $p$ is a **stable** equilibrium point.
If $L_f V(q) < 0$ for all $q \in U \setminus \{p\}$ then $p$ is an **asymptotically stable** equilibrium point.

**Theorem (Lyapunov instability theorem).** Assume that there exists an open neighbourhood $U \subset M$ of $p$ and a continuously differentiable function $V : U \to \mathbb{R}$ such that

1. $p$ is not a local minimum of the function $V$ and
2. $L_f V(q) < 0$ for all $q \in U \setminus \{p\}$.

Then $p$ is an **unstable** equilibrium point.

### First integrals

It is an important special case when the value of $V$ is constant along the trajectories.

**Definition.** The function $V$ is a **first integral** of the system $x' = f_1(x, y)$, $y' = f_2(x, y)$ if $L_f V(x, y) \equiv 0$.

**Remark.** In the special case when the planar system has the form $\begin{cases} x' = f(y) \\ y' = g(x) \end{cases}$ then by multiplying the equations we obtain $g(x) \cdot x' = f(y) \cdot y'$. Integrating both sides with respect to $t$ and then using the substitution formula ($x' \, dt = \frac{dx}{dt} dt = dx$), it follows that $\int g(x) \, dx = \int f(y) \, dy$. Thus, if $F' = f$ and $G' = g$ then $V(x, y) = F(y) - G(x)$ is a first integral.

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# Lotka-Volterra population model

Let $x(t)$ and $y(t)$ respectively denote the number of preys and predators, say, rabbits and foxes at time $t$. We assume that the prey population is the total food supply for predators. We also assume the following:

1) Preys have unlimited food supply. Hence if there were no predators, their number would grow exponentially, that is, when $y = 0$ we have $x' = a x$ where $a > 0$.

2) The prey population decreases at a rate proportional to the number of predator-prey encounters ($b x y$). So the differential equation for the prey population is $x' = a x - b x y$ where $a > 0$, $b > 0$.

3) In the absence of prey, the predator population declines at a rate proportional to the current population. So when $x = 0$ we have $y' = -c y$ with $c > 0$.

4) When there are prey in the environment, we assume that the predator population increases at a rate proportional to the predator-prey meetings, or $d x y$. Together we have $y' = -c y + d x y$ where $c > 0$, $d > 0$.

This simplified predator-prey system (also called the Volterra-Lotka system) is

$$x' = a x - b x y = x(a - b y)$$
$$y' = -c y + d x y = y(-c + d x)$$

where the parameters $a$, $b$, $c$ and $d$ are all assumed to be positive. Since we are dealing with populations, we only consider $x, y \geq 0$.

The equilibrium points are the solutions of $x(a - b y) = 0$, $y(-c + d x) = 0$.

The two solutions are $(0, 0)$ and $\left(\frac{c}{d}, \frac{a}{b}\right)$. The Jacobian matrix of the system is

$$f'(x, y) = \begin{pmatrix} a - b\,y & -b\,x \\ d\,y & -c + d\,x \end{pmatrix}.$$

At (0, 0) the Jacobian matrix is $f'(0, 0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$. The eigenvalues are $\lambda_1 = a > 0$ and $\lambda_2 = -c < 0$ and we obtain a saddle point.

At the other equilibrium point $\left(\frac{c}{d}, \frac{a}{b}\right)$, $f'\left(\frac{c}{d}, \frac{a}{b}\right) = \begin{pmatrix} 0 & -\frac{b\,c}{d} \\ \frac{d\,a}{b} & 0 \end{pmatrix}$. The eigenvalues are pure imaginary, $\lambda_{12} = \pm i\,\sqrt{a\,c}$ and so we cannot conclude anything at this stage about stability of this equilibrium point. We cannot determine the precise behavior of solutions: they could possibly spiral in toward the equilibrium point, spiral toward a limit cycle, spiral out toward "infinity" and the coordinate axes, or else lie on closed orbits. To make this determination, we search for a Lyapunov function V. Employing the trick of separation of variables, we look for a function of the form

$$V(x, y) = F(x) + G(y).$$

The derivative of $V$ with respect to the system is

$$\frac{d}{d\,t} V(x(t), y(t)) = \partial_x F \cdot x' + \partial_y G \cdot y' = \partial_x F \cdot x(a - b\,y) + \partial_y G \cdot y(-c + d\,x).$$

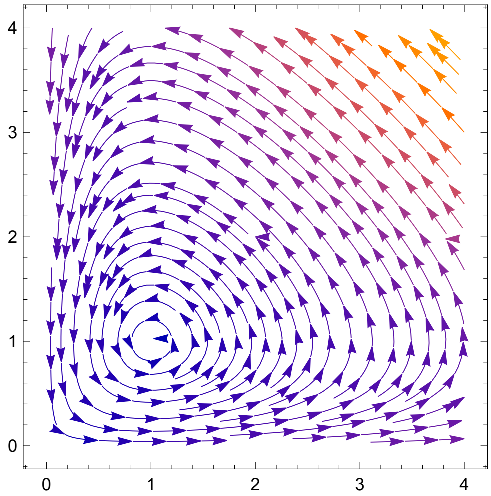
We obtain $\frac{d\,V}{d\,t} \equiv 0$ provided $\frac{\partial_x F \cdot x}{d\,x - c} = \frac{\partial_y G \cdot y}{b\,y - a}$. Since $x$ and $y$ are independent variables, this is possible if and only if $\frac{\partial_x F \cdot x}{d\,x - c} = \frac{\partial_y G \cdot y}{b\,y - a} = \text{constant}$. Setting the constant equal to 1, we obtain

$\partial_x F = d - \frac{c}{x}$ and $\partial_y G = b - \frac{a}{y}$. Integrating, we find
$F(x) = d\,x - c\log x$ and $G(y) = b\,y - a\log y$.

Thus the function $V(x, y) = d\,x - c\log x + \text{b}y - a\log y$ is constant on the solution curves of the system when $x, y > 0$.

**Theorem.** Every solution of the predator-prey system is a closed orbit (except the equilibrium point $\left(\frac{c}{d}, \frac{a}{b}\right)$ and the coordinate axes).

$\Longrightarrow$ For any given initial populations $(x(0),\ y(0))$ with $x(0) \neq 0$ and $y(0) \neq 0$, other than $\left(\frac{c}{d}, \frac{a}{b}\right)$, the populations of predator and prey oscillate cyclically. No matter what the populations of prey and predator are, neither species will die out, nor will its population grow indefinitely.




Such type of cyclic variations have been observed in nature, for example, for lynx and snowshoe hares in North Canada with a cycle of about 10 years. See also https://en.wikipedia.org/wiki/Lotka%E2%80%93Volterra_equations

**Remark.** The equation

$$x' = k_1 x - k_2 x y$$
$$y' = -k_3 y + k_2 x y$$

can be interpreted as the induced kinetic differential equation of the reaction

$$X \xrightarrow{k_1} 2X, \quad X + Y \xrightarrow{k_2} 2Y, \quad Y \xrightarrow{k_3} 0$$

and was used as a model of oscillation in cold flames by Frank-Kamenetsky in 1947. Since the Lotka–Volterra equation has a nonlinear first integral, it shows conservative oscillations, i.e. to all initial concentrations in the open first quadrant (except the unique positive stationary point) a different closed trajectory is assigned.

# Bendixson's criterion

Supplementary material.

**Theorem.** Consider the system

$$x' = f_1(x, y)$$
$$y' = f_2(x, y)$$

If in a simply-connected open set $E$ the expression $\operatorname{div} f = \partial_x f_1 + \partial_y f_2$ has constant sign (i.e. the sign remains unchanged and the expression vanishes only at isolated points or on a curve), then the above system has no closed trajectories in the set $E$.

## Exercises

1. Show that the following system does not have a periodic solution:
$x' = x + y^2 + x^3$
$y' = -x + y + x^2 y$

2. Show that following system does not have a periodic solution in the domain $x^2 + y^2 < 4$:
$x' = x - x y^2 + y^3$
$y' = 3 y - x^2 y + x^3$

3. Show that the following system does not have a periodic solution in the domain $x^2 + y^2 \leq 2/3$:
$x' = x - y - x^3$
$y' = x + y - y^3$

# Bendixson–Dulac theorem

**Theorem.** Let $E \subset \mathbb{R}^2$ be a simply-connected open set and let $B : E \to \mathbb{R}$ be a differentiable function such that $\operatorname{div}(B f) = \partial_x (B f_1) + \partial_y (B f_2)$ has constant sign and it vanishes only at isolated points or on a curve. Then the system $x' = f_1(x, y)$, $y' = f_2(x, y)$ has no periodic solutions lying entirely in the set $E$.

## Exercises

4. Consider the system $x' = x(1 - a x - b y)$
$y' = y(1 + c x - d y)$

where $a, b, c, d > 0$. Let $B(x, y) = \dfrac{1}{x y}$. Show that the system does not have a periodic solution in the set $E = \{(x, y) : x > 0, y > 0\}$.

5. Consider the system $x' = y$
$y' = x - a y + 2 x^2 + b y^2$

where $a \neq 0$. Let $B(x, y) = a e^{-2 b x}$. Show that the system does not have a periodic solution.

# Solutions

1. $f_1(x, y) = x + y^2 + x^3$, $f_2(x, y) = -x + y + x^2 y$ $\Rightarrow$
$\operatorname{div} f(x, y) = \partial_1 f_1(x, y) + \partial_2 f_2(x, y) = (1 + 3x^2) + (1 + x^2) = 2 + 4x^2 > 0$ for all $(x, y) \in \mathbb{R}^2$,
so the system does not have a periodic solution in $\mathbb{R}^2$.

2. $f_1(x, y) = x - x y^2 + y^3$, $f_2(x, y) = 3y - x^2 y + x^3$ $\Rightarrow$
$\operatorname{div} f(x, y) = \partial_1 f_1(x, y) + \partial_2 f_2(x, y) = (1 - y^2) + (3 - x^2) = 4 - (x^2 + y^2) > 0$, if $x^2 + y^2 < 4$,
so the system does not have a periodic solution in the region $x^2 + y^2 < 4$.

3. $f_1(x, y) = x - y - x^3$, $f_2(x, y) = x + y - y^3$ $\Rightarrow$
$\operatorname{div} f(x, y) = \partial_1 f_1(x, y) + \partial_2 f_2(x, y) = (1 - 3x^2) + (1 - 3y^2) = 2 - (3x^2 + 3y^2) > 0$, if $x^2 + y^2 < 2/3$,
so the system does not have a periodic solution in the region $x^2 + y^2 < 2/3$.

4. Let $f_1(x, y) = x(1 - a x - b y)$, $f_2(x, y) = y(1 + c x - d y)$.

$$B(x, y) f_1(x, y) = \frac{1}{x y} x (1 - a x - b y) = \frac{1}{y} - \frac{a x}{y} - b$$

$$\Longrightarrow \partial_1 (B(x, y) f_1(x, y)) = -\frac{a}{y}$$

$$B(x, y) f_2(x, y) = \frac{1}{x y} y(1 + c x - d y) = \frac{1}{x} + c - \frac{d y}{x}$$

$$\Longrightarrow \partial_2 (B(x, y) f_2(x, y)) = -\frac{d}{x}$$

Then $\operatorname{div}(B f)(x, y) = \partial_1 (B(x, y) f_1(x, y)) + \partial_2 (B(x, y) f_2(x, y)) = -\frac{a}{y} - \frac{d}{x} < 0$, if $x > 0$, $y > 0$,

so the system does not have a periodic solution in $E$.

5. $\operatorname{div}(B f)(x, y) = \partial_1 \left(a e^{-2bx} y\right) + \partial_2 \left(a e^{-2bx} (x - a y + 2x^2 + b y^2)\right) =$
$= \left(-2 a b e^{-2bx} y\right) + \left(-a^2 e^{-2bx} + 2 a b e^{-2bx} y\right) = -a^2 e^{-2bx} < 0$
for all $(x, y) \in \mathbb{R}^2$, so the system does not have a periodic solution in $\mathbb{R}^2$.