11 - Lyapunov's method, first integrals, exercises

1. Find a first integral for the following systems and sketch the phase portrait in the x, y plane.

   a) $x' = 5y, \; y' = -2x$ b) $x' = 1, \; y' = \sin x$

   c) $x' = 2, \; y' = 3$ d) $x' = 1, \; y' = e^x$

   e) $x' = e^y, \; y' = 2$ f) $x' = -2y, \; y' = 1$

2. Consider the equation of motion of the harmonic oscillator: $m x'' + D x = 0$ where $m > 0$, $D > 0$ are constants. Transform the equation into a linear system and find a first integral for the system. Find the energy conservation for the system and sketch the phase portrait in the x, x' plane.

3. Prove that if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function then the function $V(x, y) = x + y$ is a first integral of the system $x' = f(x, y), \; y' = -f(x, y)$.

4. The chemical reaction $X + Y \xrightarrow{k} 2X$ can be modelled be the differential equation system

$$x'(t) = k\, x(t)\, y(t)$$
$$y'(t) = -k\, x(t)\, y(t)$$

   where $x(t), y(t) \geq 0$ respectively denote the concentrations of the species X, Y and $k > 0$ is the reaction rate coefficient.

a) Find a first integral and plot the phase portrait in the x, y plane. Show that for all solutions x, y we have $\lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} x(t) > 0$. What does it mean?

b)* Calculate the solutions x and y.

5. Decide whether the function $V(x, y) = 3x - \ln x + 3y - 2\ln y$ $(x > 0, y > 0)$ is a first integral for the system $x' = 2x - 3xy, \; y' = 3xy - y$.

6. Decide whether the function $V(x, y)$ is a first integral for the system $x' = xy, \; y' = x^2 + y^2$

   if a) $V(x, y) = x \ln y - x^2 y \quad (x > 0, y > 0)$

   b) $V(x, y) = \dfrac{y^2}{x^2} - 2\ln x \quad (x > 0, y > 0)$

7. Using the Lyapunov function $V(x, y) = 2x^2 + 3y^2$, investigate the stability of the origin in the system $x' = -3y - x^5, \quad y' = 2x - y^5$.

8. Investigate the stability of the origin in the following systems using the Lyapunov function $V(x, y) = a x^2 + b y^2$ where $a > 0$, $b > 0$.

   a) $x' = y - x^3$, $y' = -x - y^3$

   b) $x' = -2y + x^5$, $y' = x + y^5$

   c) $x' = -y + x^3$, $y' = x + y^3$

   d) $x' = 3y - x$, $y' = -2x - y$

Solutions

1. Find a first integral for the following systems and sketch the phase portrait in the x, y plane.
   a) $x' = 5y, \ y' = -2x$

Solution.

• The product of the equations: $-2x \cdot x' = 5y \cdot y'$

Integrating both sides with respect to the time: $\int -2x \cdot x' \, dt = \int 5y \cdot y' \, dt$

Using the substitution formula ($x' \, dt = \frac{dx}{dt} dt = dx$): $\int -2x \, dx = \int 5y \, dy \implies -x^2 = \frac{5y^2}{2} + c$

$\implies$ The function $V(x, y) = x^2 + \frac{5y^2}{2}$ is a first integral for the system.

Geometrically it means that the trajectories go along the nullclines of V.

• Verification: $\frac{d}{dt} V(x(t), y(t)) = 2x \cdot x' + \frac{5}{2} \cdot 2y \cdot y' = 2x \cdot 5y + \frac{5}{2} \cdot 2y \cdot (-2x) = 0$

• Remark: It is obvious that for example $V(x, y) = -\left(x^2 + \frac{5y^2}{2}\right)$ or

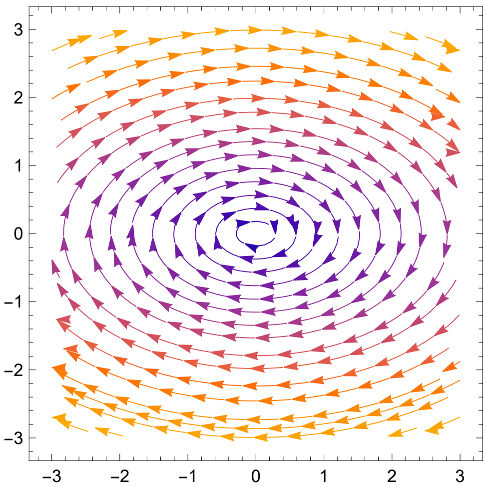
$V(x, y) = 3\left(x^2 + \frac{5y^2}{2}\right) + 16$ are also first integrals.

• The equation of the trajectories: $x^2 + \frac{5y^2}{2} = C$ (ellipses centered at the origin)

• Direction of the trajectories: for example at (1, 1): $x' = 5 > 0 \implies x$ increases

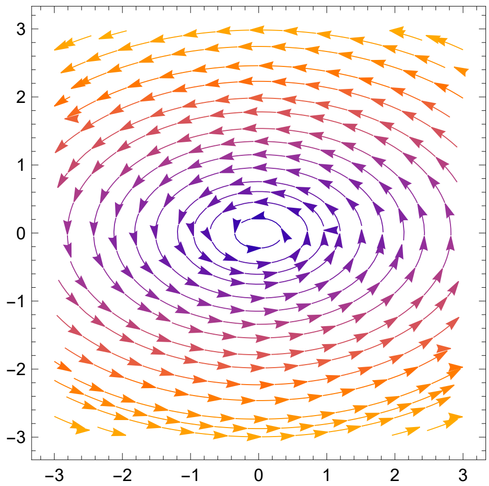
$y' = -2 < 0 \implies y$ decreases

$\implies$ the trajectories go clockwise along the ellipses



• Remark: Similarly as above, the function $V(x, y) = x^2 + \frac{5y^2}{2}$ is also a first integral for the system $x' = -5y, \ y' = 2x$, but here the trajectories go anticlockwise along the ellipses, since for example at (1, 1): $x' = -5 < 0 \implies x$ decreases

$y' = 2 > 0 \implies y$ increases



b) $x' = 1, \; y' = \sin x$

Solution.

• The product of the equations: $\sin x \cdot x' = 1 \cdot y'$

Integrating both sides with respect to the time: $\int \sin x \cdot x' \, \mathrm{dt} = \int 1 \cdot y' \, \mathrm{dt}$

Using the substitution formula ($x' \, \mathrm{dt} = \frac{dx}{dt} \, dt = dx$): $\int \sin x \, \mathrm{dx} = \int 1 \, \mathrm{dy} \implies -\cos x = y + c$

$\implies$ The function $V(x, y) = \cos x + y$ is a first integral for the system

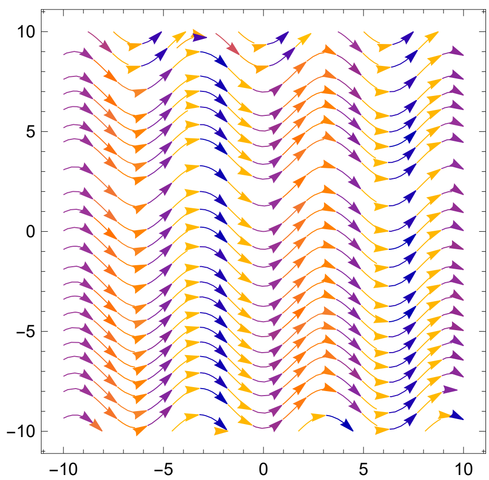
• Verification: $\frac{d}{\mathrm{dt}} V(x(t), y(t)) = -\sin x \cdot x' + 1 \cdot y' = -\sin x \cdot 1 + 1 \cdot \sin x = 0$

• The equation of the trajectories: $\cos x + y = C$ or $y = -\cos x + C$

• Direction of the trajectories: $x' = 1 > 0 \implies x$ increases $\implies$ the trajectories go to the right

Remark: $y' = \sin x > 0$ if $2 k \pi < x < \pi + 2 k \pi$ (k is an integer) $\implies y$ increases

$y' = \sin x < 0$ if $\pi + 2 k \pi < x < 2 \pi + 2 k \pi$ (k is an integer) $\implies y$ decreases



c) $x' = 2, \; y' = 3$

Solution.

• The product of the equations: $3\cdot x' = 2\cdot y'$

Integrating both sides with respect to the time: $\int 3\cdot x'\,\mathrm{dt} = \int 2\cdot y'\,\mathrm{dt}$

Using the substitution formula ($x'\,\mathrm{dt} = \frac{dx}{dt}\,dt = dx$): $\int 3\,\mathrm{dx} = \int 2\,\mathrm{dy} \implies 3x = 2y + c$

$\implies$ The function $V(x, y) = 3x - 2y$ is a first integral for the system

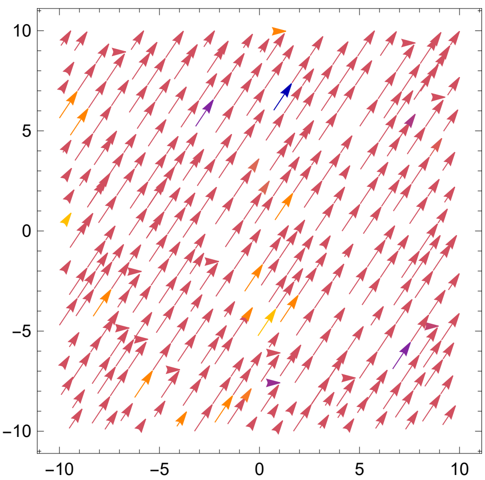
• Verification: $\frac{d}{\mathrm{dt}} V(x(t), y(t)) = 3\cdot x' - 2\cdot y' = 3\cdot 2 - 2\cdot 3 = 0$

• Remark: $V(x, y) = 2y - 3x$ or $V(x, y) = 5(3x - 2y) + 7$ is also a first integral.

• The equation of the trajectories: $3x - 2y = C$ (straight lines)

Direction of the trajectories: $x' = 2 > 0 \implies x$ increases

$y' = 3 > 0 \implies y$ increases



d) $x' = 1,\ \ y' = e^x$

Solution.

• The product of the equations: $e^x\cdot x' = 1\cdot y'$

Integrating both sides with respect to the time: $\int e^x\cdot x'\,\mathrm{dt} = \int 1\cdot y'\,\mathrm{dt}$

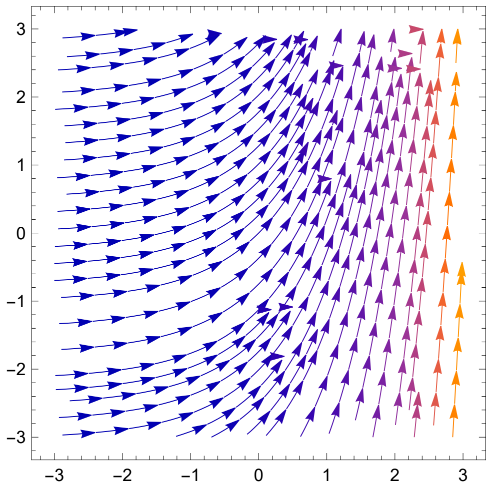
$\implies \int e^x\,\mathrm{dx} = \int 1\,\mathrm{dy} \implies e^x = y + c$

$\implies$ The function $V(x, y) = y - e^x$ is a first integral for the system.

• Nullclines: $y = e^x + C$, that is, these are vertical shifts of the graph of $y = e^x$.

• Direction of the trajectories: for example at (1, 1): $x' = 1 > 0 \implies x$ increases

$y' = e > 0 \implies y$ increases



e) $x' = e^y,\ y' = 2$

Solution.

• The product of the equations: $2 \cdot x' = e^y \cdot y'$

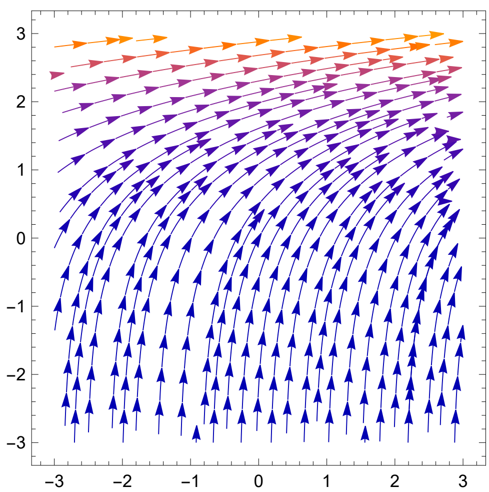
Integrating both sides with respect to the time: $\int 2 \cdot x' \, dt = \int e^y \cdot y' \, dt$

$\Longrightarrow \int 2 \, dx = \int e^y \, dy \Longrightarrow 2x = e^y + c$

$\Longrightarrow$ The function $V(x, y) = x - \frac{1}{2} e^y$ is a first integral for the system.

• Nullclines: $x = \frac{1}{2} e^y + C$, that is, these are horizontal shifts of the graph of $x = \frac{1}{2} e^y$ (or $y = \ln(2x)$)

• Direction of the trajectories: for example at (1, 1): $x' = e > 0 \Longrightarrow x$ increases
$y' = 2 > 0 \Longrightarrow y$ increases



f) $x' = -2y,\ y' = 1$

Solution.

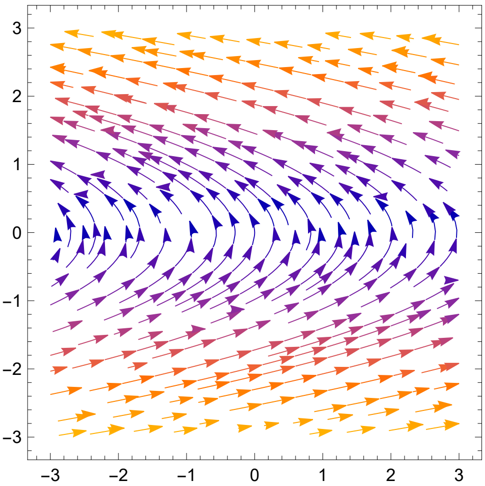
• The product of the equations: $1 \cdot x' = -2y \cdot y'$

Integrating both sides with respect to the time: $\int 1 \cdot x' \, dt = \int -2y \cdot y' \, dt$

$\Longrightarrow \int 1\, dx = \int -2\, y\, dy \Longrightarrow x = -y^2 + c$

$\Longrightarrow$ The function $V(x, y) = x + y^2$ is a first integral for the system.

- Nullclines: $x = -y^2 + c$, that is, these are horizontal shifts of the graph of $x = -y^2$ (horizontal parabolas).
- Direction of the trajectories: Since $y' = 1 > 0$ then y increases.
  If $y > 0$ (above the x axis) then $x' = -2\, y < 0 \Longrightarrow x$ decreases.
  If $y < 0$ (below the x axis) then $x' = -2\, y > 0 \Longrightarrow x$ increases.



2. Consider the equation of motion of the harmonic oscillator: $m\, x'' + D\, x = 0$ where $m > 0$, $D > 0$ are constants. Transform the equation into a linear system and find a first integral for the system. Find the energy conservation for the system and sketch the phase portrait in the x, x' plane.

Solution. From the equation $m\, x'' + D\, x = 0 \Longrightarrow x'' = -\frac{D}{m}\, x$.

We apply the following substitution.

Let $x_1 = x \quad \Longrightarrow x_1' = x' = x_2 \qquad \Longrightarrow x_1' = x_2 \qquad$ or: $x' = y$

$x_2 = x' \qquad x_2' = x'' = -\frac{D}{m}\, x = -\frac{D}{m}\, x_1 \qquad x_2' = -\frac{D}{m}\, x_1 \qquad y' = -\frac{D}{m}\, x$

The product of the equations: $-\frac{D}{m}\, x \cdot x' = y \cdot y'$

Integrating both sides with respect to the time: $\int -\frac{D}{m}\, x\, dx = \int y\, dy \Longrightarrow -\frac{D}{m}\, \frac{x^2}{2} = \frac{y^2}{2} + c$

The function $V(x, y) = \frac{1}{2}\, D\, x^2 + \frac{1}{2}\, m\, y^2$ is a first integral for the system.

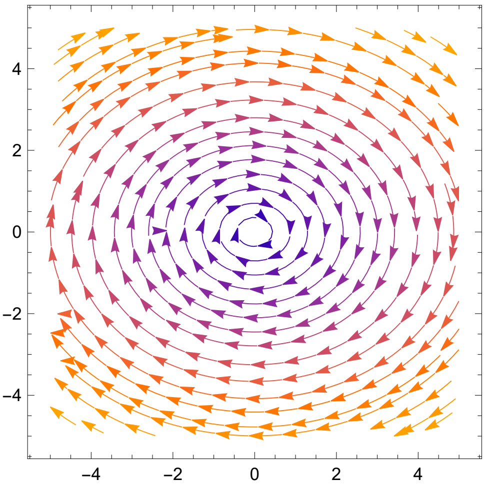
The physical meaning of the terms is the following:

- $\frac{1}{2}\, D\, x^2$ is the potential energy (x is the displacement) and
- $\frac{1}{2}\, m\, y^2$ is the kinetic energy ($y = x'$ is the velocity)

$\Longrightarrow V$ is total mechanical energy of the system.

The equation of the trajectories: $\frac{1}{2}\, D\, x^2 + \frac{1}{2}\, m\, y^2 = C$ (ellipses centered at the origin)

The phase portrait for $m = 3$, $d = 2$:

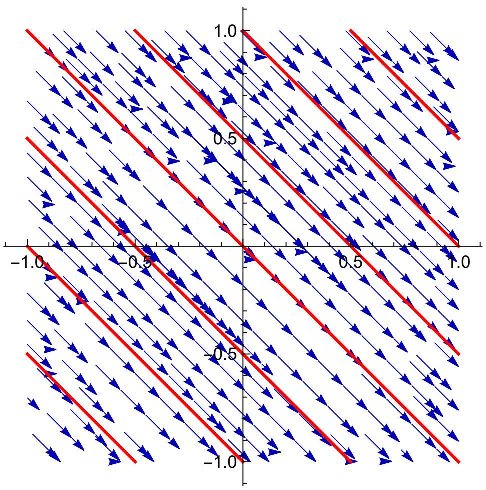


3. Prove that if $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function then the function $V(x, y) = x + y$ is a first integral of the system $x' = f(x, y)$, $y' = -f(x, y)$.

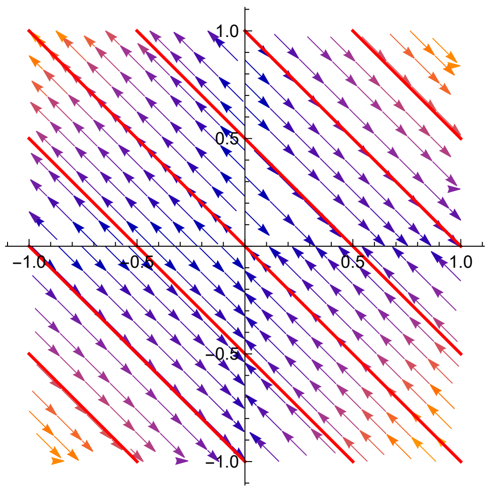
Solution. Adding the two equations $(x + y)' = 0$, so $x(t) + y(t) = \text{constant} \implies V(x, y) = x + y$ is a first integral,
since $(V(x(t), y(t)))' = x'(t) + y'(t) = 0$.
The equation of the trajectories: $x + y = c \implies y = c - x$, so the trajectories are straight lines.

The direction of the trajectories can be determined from the sign of the derivatives, as the following examples show.

(a) Phase portrait of the system
$x' = 1,\ y' = -1$



(b) Phase portrait of the system
$x' = x\,y,\ y' = -x\,y$



Case (a): $x' > 0,\ y' < 0 \implies x$ increases, y decreases.
Case (b):
In the 1st quadrant $x > 0, y > 0 \implies x' > 0,\ y' < 0 \implies x$ increases, y decreases.
In the 2nd quadrant $x < 0, y > 0 \implies x' < 0,\ y' > 0 \implies x$ decreases, y increases.
In the 3rd quadrant $x < 0, y < 0 \implies x' > 0,\ y' < 0 \implies x$ increases, y decreases.
In the 4th quadrant $x < 0, y > 0 \implies x' < 0,\ y' > 0 \implies x$ decreases, y increases.

4. The chemical reaction $X + Y \xrightarrow{k} 2X$ can be modelled be the differential equation system

$$x'(t) = k\,x(t)\,y(t)$$
$$y'(t) = -k\,x(t)\,y(t)$$

where $x(t)$, $y(t) \geq 0$ respectively denote the concentrations of the species X, Y and $k > 0$ is the reaction rate coefficient.

a) Find a first integral and plot the phase portrait in the x, y plane. Show that for all solutions x, y we have $\lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} x(t) > 0$. What does it mean?

b)* Calculate the solutions x and y.

Solution. a) Similarly to the previous exercise, adding the two equations we have $(x(t) + y(t))' = 0$, so $x(t) + y(t) = \text{constant} \implies V(x, y) = x + y$ is a first integral. It means that the trajectories are straight line segments in the first quadrant with equations $y = c - x$ where $c \geq 0$. (The chemical meaning of this is that the total amount of the two substances remains constant.)

The direction of the trajectories on the straight lines can be determined by the signs of the derivatives. Since $x' > 0$ and $y' < 0$ then x increases and y decreases along the trajectories. So from the phase portrait it can be seen that $x(t) \longrightarrow x_0 > 0$ and $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$. It means that after a long time the concentration of the first species will roughly reach a constant value while the concentration of the second one will be approximately zero.

The points $(x(t), y(t))$ will never reach the x axis, since the points of this axis correspond to the constant solutions $(x_0, 0)$ and by the uniqueness of the solutions, the trajectories don't intersect each other.

b) Since $x(t) + y(t) = M$ (constant) then $y(t) = M - x(t)$. Substituting it into the first equation we get $x'(t) = k\,x(t)\,(M - x(t))$, which is the equation of the logistic population model (see exercises 02-8 and 05-5).

5. Decide whether the function $V(x, y) = 3x - \ln x + 3y - 2\ln y$ $(x > 0, y > 0)$ is a first integral for the system $x' = 2x - 3xy$, $y' = 3xy - y$.

Solution.

a) The derivative of $V(x, y) = 3x - \ln x + 3y - 2\ln y$ with respect to the system $x' = 2x - 3xy$, $y' = 3xy - y$ is:

$$\frac{d}{dt} V(x(t), y(t)) = 3x' - \frac{1}{x}\cdot x' + 3y' - 2\cdot\frac{1}{y}\cdot y' =$$

$$= 3(2x - 3xy) - \frac{1}{x}\cdot(2x - 3xy) + 3(3xy - y) - 2\cdot\frac{1}{y}\cdot(3xy - y) =$$

$$= 6x - 9xy - 2 + 3y + 9xy - 3y - 6x + 2 \equiv 0$$

Since the Lie derivative of V is identically zero then V is a first integral for the system.

6. Decide whether the function $V(x, y)$ is a first integral for the system $x' = x\,y,\quad y' = x^2 + y^2$
if a) $V(x, y) = x \ln y - x^2 y \quad (x > 0, y > 0)$
b) $V(x, y) = \frac{y^2}{x^2} - 2\ln x \quad (x > 0, y > 0)$

Solution.
a) The derivative of $V(x, y) = x \ln y - x^2 y$ with respect to the system $x' = x\,y,\quad y' = x^2 + y^2$ is:

$$L_f V(x, y) = \frac{d}{dt} V(x(t), y(t)) = x' \ln y + x \ln y \cdot y' - 2\,x\,x'\,y - x^2\,y' =$$
$$= x\,y \ln y + x \ln y \cdot (x^2 + y^2) - 2\,x\,x\,y\,y - x^2\,(x^2 + y^2)$$

This expression is not identically zero, for example if $(x, y) = (1, 1)$ then
$L_f V(x, y) = -4 \neq 0 \implies V$ is not a first integral for the system.

b) The derivative of $V(x, y) = \frac{y^2}{x^2} - 2\ln x = x^{-2} y^2 - 2\ln x$ with respect to the system
$x' = x\,y,\quad y' = x^2 + y^2$ is:

$$L_f V(x, y) = -2\,x^{-3}\,x'\,y^2 + x^{-2}\,2\,y\,y' - 2 \cdot \frac{1}{x} \cdot x' =$$
$$= -2\,x^{-3}\,x\,y\,y^2 + x^{-2}\,2\,y\,(x^2 + y^2) - 2 \cdot \frac{1}{x} \cdot x\,y =$$
$$= -2\,x^{-2}\,y^3 + 2\,y + 2\,x^{-2}\,y^3 - 2\,y \equiv 0$$

Since the Lie derivative of V is identically zero then V is a first integral for the system.

7. Using the Lyapunov function $V(x, y) = 2\,x^2 + 3\,y^2$, investigate the stability of the origin in the system $x' = -3\,y - x^5,\quad y' = 2\,x - y^5$.

Solution.
The Lyapunov function: $V(x, y) = 2\,x^2 + 3\,y^2 \implies$ the nullclines of V are ellipses
centered at the origin: $2\,x^2 + 3\,y^2 = c$

The system: $x' = -3\,y - x^5,\quad y' = 2\,x - y^5$
The derivative of V with respect to the system is:

$$L_f V(x, y) = \frac{d}{dt} V(x(t), y(t)) = V'(x, y) \cdot f(x, y) = \partial_x V(x, y) \cdot f_1(x, y) + \partial_y V(x, y) \cdot f_2(x, y) =$$
$$= 4\,x \cdot x' + 6\,y \cdot y' = 4\,x\,(-3\,y - x^5) + 6\,y\,(2\,x - y^5) = -12\,x\,y - 6\,x^6 + 12\,x\,y - 6\,y^6 =$$
$$= -6\,(x^6 + y^6) < 0 \quad \text{if } (x, y) \neq (0, 0)$$

$\implies$ the value of V decreases along the trajectories, so the trajectories go towards the origin (they intersect the nullclines going inwards)
$\implies$ the origin is asymptotically stable

8. Investigate the stability of the origin in the following system using the Lyapunov function $V(x, y) = a\,x^2 + b\,y^2$ where $a > 0$, $b > 0$.

a) $x' = y - x^3,\quad y' = -x - y^3$

Solution.
The Lyapunov function: $V(x, y) = a\,x^2 + b\,y^2$ where $a > 0$, $b > 0$

$\Longrightarrow$ the nullclines of V are ellipses centered at the origin

The system: $x' = y - x^3$, $y' = -x - y^3$

The derivative of V with respect to the system is:

$$\frac{d}{dt} V(x(t), y(t)) = 2\,a\,x \cdot x' + 2\,b\,y \cdot y' = 2\,a\,x\left(y - x^3\right) + 2\,b\,y\left(-x - y^3\right) =$$

$$= -2\left(a\,x^4 + b\,y^4\right) + 2\,(a - b)\,x\,y$$

Here the first term is always negative, however, the sign of $x\,y$ can be arbitrary.
So it is worth choosing a and b such that $a - b = 0$, for example let $a = 1$, $b = 1$.

Then for the Lyapunov function $V(x, y) = x^2 + y^2$ we have

$$\frac{d}{dt} V(x(t), y(t)) = -2\left(x^4 + y^4\right) < 0 \quad \text{if } (x, y) \neq (0, 0)$$

$\Longrightarrow$ the value of V decreases along the trajectories, so the trajectories go towards the origin (they intersect the nullclines going inwards)

$\Longrightarrow$ the origin is asymptotically stable

8. Investigate the stability of the origin in the following system using the Lyapunov function $V(x, y) = a\,x^2 + b\,y^2$ where $a > 0$, $b > 0$.

   b) $x' = -2\,y + x^5$, $\quad y' = x + y^5$

Solution.

The Lyapunov function: $V(x, y) = a\,x^2 + b\,y^2$ where $a > 0$, $b > 0$

$\Longrightarrow$ the nullclines of V are ellipses centered at the origin

The system: $x' = -2\,y + x^5$, $y' = x + y^5$

The derivative of V with respect to the system is:

$$\frac{d}{dt} V(x(t), y(t)) = 2\,a\,x \cdot x' + 2\,b\,y \cdot y' = 2\,a\,x\left(-2\,y + x^5\right) + 2\,b\,y\left(x + y^5\right) =$$

$$= 2\left(a\,x^6 + b\,y^6\right) + (-4\,a + 2\,b)\,x\,y$$

We choose a and b such that $-4\,a + 2\,b = 0$, for example let $a = 1$, $b = 2$.

Then for the Lyapunov function $V(x, y) = x^2 + 2\,y^2$ we have

$$\frac{d}{dt} V(x(t), y(t)) = 2\,x^6 + 4\,y^6 > 0 \quad \text{if } (x, y) \neq (0, 0)$$

$\Longrightarrow$ the value of V increases along the trajectories, so the trajectories go away from the origin (they intersect the nullclines going outwards)

$\Longrightarrow$ the origin is unstable