10 - Nonlinear systems

Introduction

Consider the following n-dimensional autonomous system:

(1) $x'(t) = f(x(t))$

This equation usually cannot be solved by a formula so it is the phase portrait that provides most information about the solutions.

We obtain the constant solutions $x(t) \equiv p$ by solving the algebraic equation system $f(p) = 0$. These constant solutions are also called **equilibrium points** or **stationary points**. The behaviour of the trajectories in a small neighbourhood of the equilibrium points can be determined by **linearization**. In order to understand that, introduce the function $y(t) = x(t) - p$. Then the differential equation for y is the following:

$$y'(t) = x'(t) = f(x(t)) = f(p) + f'(p)\,y(t) + r(y(t)) = 0 + f'(p)\,y(t) + r(y(t))$$

where r denotes the remainder term. Since for a small y the remainder term is smaller in magnitude than the linear term (if the linear term is not too small, for example zero) then it can be expected that in a small neighbourhood of the equilibrium point p, the phase portrait is determined by the linearized equation

(2) $y'(t) = f'(p)\,y(t)$

where $f'(p)$ is called the **Jacobian matrix**. We will clarify what it means that the linear term is not too small and in what sense the phase portrait is determined by the linear term. This is described by the following notions and theorems.

Definition of stability

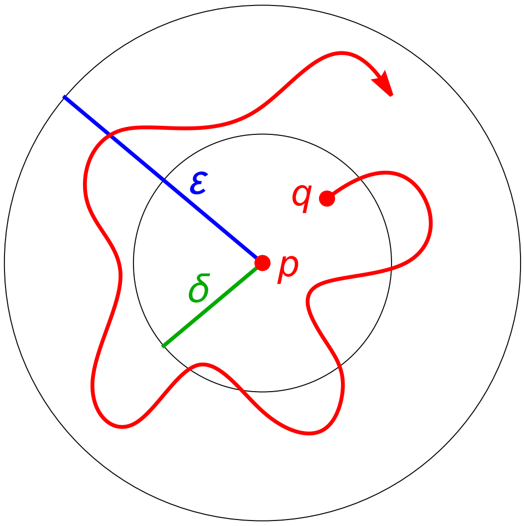
Definition. 1) Let $t \mapsto \varphi(t, p)$ (where $p \in \mathbb{R}^n$) denote the solution of (1) for which the initial condition $\varphi(0) = p$ holds. Then p is called a **stable equilibrium point** of (1) if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $q \in D_f$,

$$|q - p| < \delta \implies |\varphi(t, q) - p| < \varepsilon \quad \text{if } t \geq 0.$$

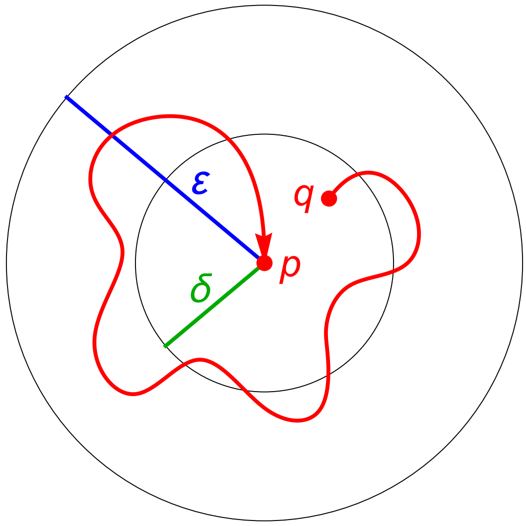
2) The equilibrium point p is called **asymptotically stable** if it is stable and for the above choice of q, $\lim_{t\to\infty} |\varphi(t, q) - p| = 0$.

3) The equilibrium point p is called **unstable** if it is not stable.

Stable equilibrium point



Asymptotically stable equilibrium point



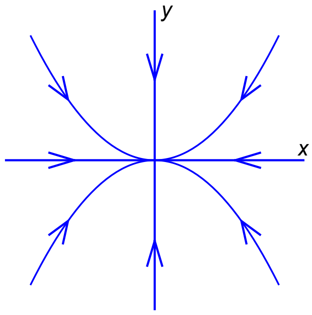
Remark. The **stability** of the equilibrium point p means that if the distance of the initial point q from p is less than δ then the points of the trajectory starting from q remain within a distance ε from p. In other words, all trajectories that are initially sufficiently close to p, remain close to p for all future times.

The **asymptotic stability** of p means that all trajectories that are initially within a distance δ of p, tend towards p as $t \to \infty$.

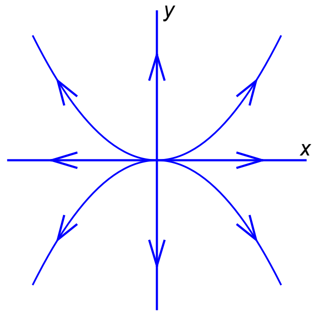
Question: Are the following type of equilibrium points stable, asymptotically stable or unstable?
(1) real sink, (2) real source, (3) saddle, (4) spiral sink, (5) spiral source, (6) center.

Solution:

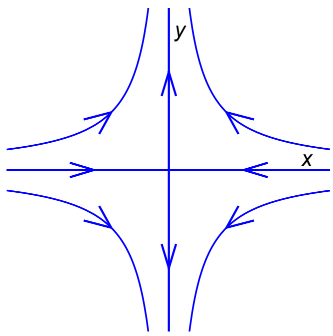
1) sink: asymptotically stable



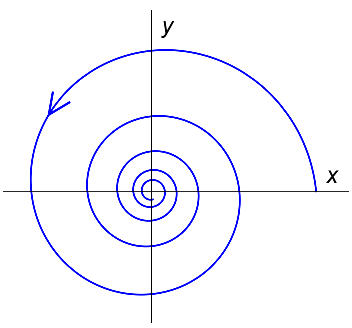
2) source: unstable



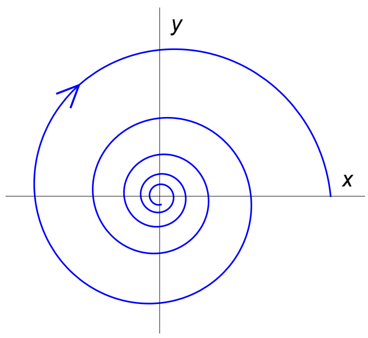
3) saddle: unstable



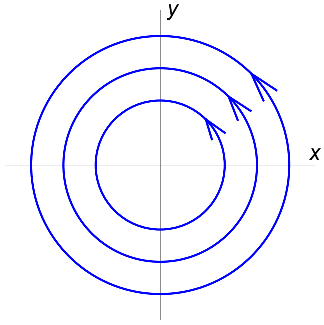
4) spiral sink: asymptotically stable



5) spiral source: unstable



6) center: stable, but not asymptotically stable



Linearization

By linearization, stability can be determined in the following way.

Theorem. 1) If every eigenvalue of $A = f'(p)$ has negative real part then p is an asymptotically stable equilibrium point of system (1).

2) If any of the eigenvalues of $A = f'(p)$ has positive real part then p is an unstable equilibrium point of system (1).

Planar systems

The phase portrait of planar systems can be characterized more precisely. For the characterization we define the type of the equilibrium point for nonlinear systems based on the geometrical properties of the types of linear systems.

Definition. Consider the two-dimensional autonomous system $x' = f_1(x, y)$, $y' = f_2(x, y)$. In a neighbourhood U of the equilibrium point p, let us represent the solutions in polar coordinates r, φ: $x(t) = r(t) \cdot \cos(\varphi(t))$, $y(t) = r(t) \cdot \sin(\varphi(t))$. Then the point p is a

(1) **real sink** if $\lim_{t \to +\infty} r = 0$ and $\lim_{t \to +\infty} |\varphi| < \infty$;

(2) **real source** if $\lim_{t \to -\infty} r = 0$ and $\lim_{t \to -\infty} |\varphi| < \infty$;

(3) **saddle** if there exist two trajectories in U that tend towards p as $t \to +\infty$, there exist two trajectories in U that tend towards p as $t \to -\infty$, and all the other trajectories starting from a point different from p, leave the neighbourhood U as $t \to +\infty$ and as $t \to -\infty$.

(4) **spiral sink** if $\lim_{t \to +\infty} r = 0$ and $\lim_{t \to +\infty} |\varphi| = \infty$;

(5) **spiral source** if $\lim_{t \to -\infty} r = 0$ and $\lim_{t \to -\infty} |\varphi| = \infty$;

(6) **center** if every trajectory in U is periodic (except the equilibrium point p).

Remark: $\lim_{t \to \pm\infty} |\varphi| < \infty$ means the the trajectories go around p finitely many times.

$\lim_{t \to \pm\infty} |\varphi| = \infty$ means that the trajectories go around p infinitely many times.

Definition. The equilibrium point p is called **hyperbolic** if $\mathrm{Re}(\lambda) \neq 0$ for each eigenvalue λ of the Jacobian matrix $f'(p)$.

Theorem. Let $n = 2$. If p is a hyperbolic equilibrium point of the nonlinear system $x'(t) = f(x(t))$ then it has the same type as the origin of the linearized system $y'(t) = f'(p)\,y(t)$.

Remark. The Lyapunov method plays an important role in the local investigation of non-hyperbolic equilibrium points. This method can also be used in the global investigation of the phase portrait. See Lectures notes 11.