10 - Nonlinear systems, exercises

In the following nonlinear systems, find all of the equilibrium points and describe the behavior of the associated linearized systems at the equilibrium points.

1. $x' = x - y$, $y' = x + y - 2xy$
2. $x' = 1 - xy$, $y' = (x-1)y$
3. $x' = (1 + x - 2y)x$, $y' = (x-1)y$
4. $x' = x - y$, $y' = x^2 - 1$
5. $x' = -6y + 2xy - 8$, $y' = y^2 - x^2$

Solutions

1. $x' = x - y,\ y' = x + y - 2xy$

Solution. The system

$$\begin{aligned} x' &= x - y &&= f_1(x,y) \\ y' &= x + y - 2xy &&= f_2(x,y) \end{aligned}$$

can be written as $X'(t) = f(X(t))$ where $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x - y \\ x + y - 2xy \end{pmatrix}$

The equilibrium (or stationary or singular) points of the system:

$$\begin{aligned} x' &= x - y = 0 && \Longrightarrow x = y && \Longrightarrow x_1 = 0,\ y_1 = 0 \quad \text{or} \\ y' &= x + y - 2xy = 0 && 2x - 2x^2 = 2x(1-x) = 0 && x_2 = 1,\ y_2 = 1 \end{aligned}$$

The equilibrium points are: $P_1(0, 0)$ and $P_2(1, 1)$.

The Jacobian of the linearized system:

$$f'(x,y) = \begin{pmatrix} \dfrac{\partial f_1(x,y)}{\partial x} & \dfrac{\partial f_1(x,y)}{\partial y} \\ \dfrac{\partial f_2(x,y)}{\partial x} & \dfrac{\partial f_2(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 - 2y & 1 - 2x \end{pmatrix}$$

The eigenvalues of the Jacobian at the equilibrium points are the following.

Case 1.

At $P_1(0, 0)$ the Jacobian is $f'(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\Longrightarrow \lambda_{1,2} = 1 \pm i$ (complex eigenvalues with positive real part)

$\Longrightarrow P_1(0, 0)$ is a spiral source

Or: trace$(f'(0, 0)) = T = 1 + 1 = 2 > 0$

det$(f'(0, 0)) = D = 1 - (-1) = 2 > 0$

$T^2 - 4D = 4 - 4 \cdot 2 = -4 < 0$

$\Longrightarrow P_1(0, 0)$ is a spiral source

Case 2.

At $P_2(1, 1)$ the Jacobian is $f'(1, 1) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$

$\Longrightarrow \lambda_1 = \sqrt{2},\ \lambda_2 = -\sqrt{2}$ (real eigenvalues of the opposite sign)

$\Longrightarrow P_2(1, 1)$ is a saddle

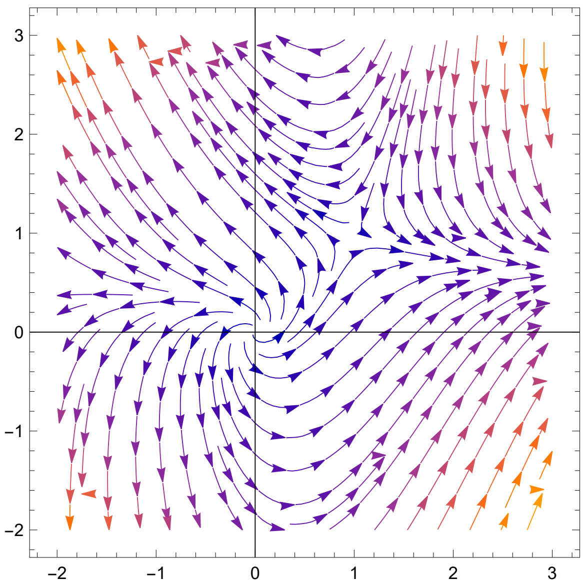
Or: trace$(f'(1, 1)) = T = 1 - 1 = 0$

det$(f'(1, 1)) = D = -1 - 1 = -2 < 0$

$T^2 - 4D = 0 - 4 \cdot (-2) = 8 > 0$

$\Longrightarrow P_2(1, 1)$ is a saddle

The phase portrait of the system:



2. $x' = 1 - xy,\ y' = (x - 1)y$

Solution.

$x' = 1 - xy \quad = f_1(x, y)$

$y' = (x - 1)y = f_2(x, y)$

The equilibrium (or stationary or singular) points of the system:

$x' = 1 - x\,y = 0 \quad \Longrightarrow x = 1 \text{ or } y = 0 \Longrightarrow x = 1,\ y = 1$

$y' = (x - 1)\,y = 0$

(if $y = 0$ then we get a contradiction from the first equation)

The only equilibrium point is: $P(1, 1)$.

The Jacobian of the linearized system:

$$f'(x, y) = \begin{pmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} -y & -x \\ y & x-1 \end{pmatrix}$$

The eigenvalues of the Jacobian at the equilibrium point are the following.

At $P(1, 1)$ the Jacobian is $f'(1, 1) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

$\Longrightarrow \lambda_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ (complex eigenvalues with negative real part)

$\Longrightarrow P(1, 1)$ is a spiral sink.

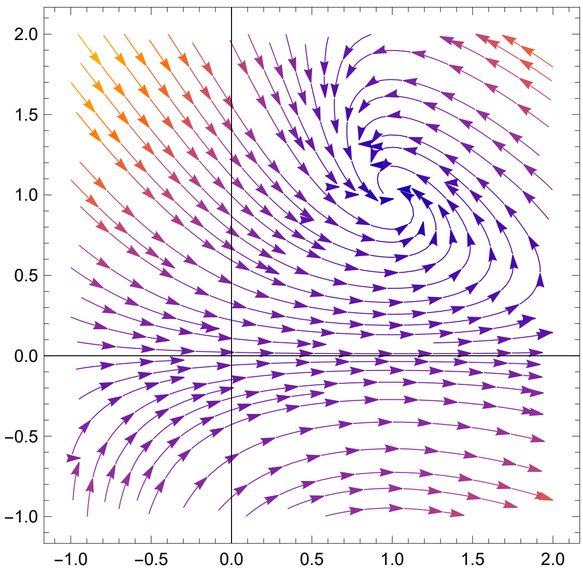
Or: $\text{trace}(f'(1, 1)) = T = -1 + 0 = -1 < 0$

$\det(f'(1, 1)) = D = 0 - (-1) = 1 > 0$

$T^2 - 4\,D = 1 - 4 \cdot 1 = -3 < 0$

$\Longrightarrow P(1, 1)$ is a spiral sink.

The phase portrait of the system:



3. $x' = (1 + x - 2y)x,\ y' = (x - 1)y$

Solution.

$x' = (1 + x - 2y)x = f_1(x, y)$

$y' = (x - 1)y = f_2(x, y)$

The equilibrium (or stationary or singular) points of the system:

$x' = (1 + x - 2y)x = 0 \implies x = 1$ or $y = 0$

$y' = (x - 1)y = 0$

If $x = 1$: $2 - 2y = 0 \implies y = 1$

If $y = 0$: $(1 + x)x = 0 \implies x = 0$ or $x = -1$

The equilibrium points are: $P_1(1, 1)$, $P_2(0, 0)$ and $P_3(-1, 0)$.

The Jacobian of the linearized system:

$x' = (1 + x - 2y)x = x + x^2 - 2xy = f_1(x, y)$

$y' = (x - 1)y = xy - y \qquad\qquad = f_2(x, y)$

$$f'(x, y) = \begin{pmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 + 2x - 2y & -2x \\ y & x - 1 \end{pmatrix}$$

The eigenvalues of the Jacobian at the equilibrium points are the following.

Case 1.

At $P_1(1, 1)$ the Jacobian is $f'(1, 1) = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$

$\implies \lambda_{1,2} = \frac{1}{2} \pm \frac{i\sqrt{7}}{2}$ (complex eigenvalues with positive real part)

$\implies P_1(1, 1)$ is a spiral source

Case 2.

At $P_2(0, 0)$ the Jacobian is $f'(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\implies \lambda_1 = 1,\ \lambda_2 = -1$ (real eigenvalues of the opposite sign)

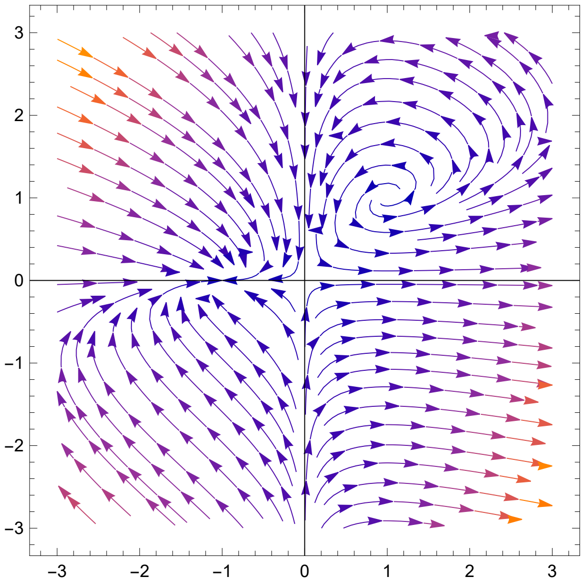
$\implies P_2(0, 0)$ is a saddle

Case 3.

At $P_3(-1, 0)$ the Jacobian is $f'(-1, 0) = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix}$

$\Longrightarrow \lambda_1 = -1,\ \lambda_2 = -2$ (real negative eigenvalues)
$\Longrightarrow P_3(-1, 0)$ is a real sink

The phase portrait of the system:



4. $x' = x - y,\ y' = x^2 - 1$

Solution.
$x' = x - y \ = f_1(x, y)$
$y' = x^2 - 1 = f_2(x, y)$

The equilibrium (or stationary or singular) points of the system:
$x' = x - y = 0 \quad \Longrightarrow x = 1$ or $x = -1$
$y' = x^2 - 1 = 0 \qquad x = y$

The equilibrium points are: $P_1(1, 1)$, and $P_2(-1, -1)$.

The Jacobian of the linearized system:

$$f'(x, y) = \begin{pmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$$

The eigenvalues of the Jacobian at the equilibrium points are the following.

Case 1.
At $P_1(1, 1)$ the Jacobian is $f'(1, 1) = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$

$\Longrightarrow \lambda_{1,2} = \frac{1}{2} \pm \frac{i\sqrt{7}}{2}$ (complex eigenvalues with positive real part)

$\Longrightarrow P_1(1, 1)$ is a spiral source

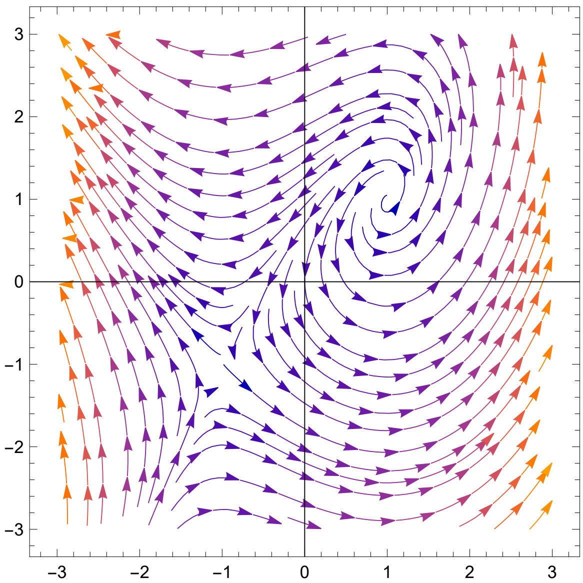
Case 2.

At $P_2(-1, -1)$ the Jacobian is $f'(-1, -1) = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

$\Longrightarrow \lambda_1 = 2,\ \lambda_2 = -1$ (real eigenvalues of the opposite sign)

$\Longrightarrow P_2(-1, -1)$ is a saddle

The phase portrait of the system:



5. $x' = -6y + 2xy - 8,\ y' = y^2 - x^2$

Solution.

$x' = -6y + 2xy - 8 = f_1(x, y)$

$y' = y^2 - x^2 \qquad\qquad = f_2(x, y)$

The equilibrium (or stationary or singular) points of the system:

$x' = -6y + 2xy - 8 = 0 \Longrightarrow x = y$ or $x = -y$

$y' = y^2 - x^2 = 0$

If $x = y$: $2x^2 - 6x - 8 = 0 \Longrightarrow x^2 - 3x - 4 = (x - 4)(x + 1) = 0 \Longrightarrow x = -1$ or $x = 4$

If $x = -y$: $-2x^2 + 6x - 8 = 0 \Longrightarrow x^2 - 3x + 4 = 0 \Longrightarrow x_{1,2} = \frac{1}{2}\left(3 \pm i\sqrt{7}\right)$ this cannot be the case

The equilibrium points are: $P_1(-1, -1)$, and $P_2(4, 4)$.

The Jacobian of the linearized system:

$$f'(x,y) = \begin{pmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 2y & -6+2x \\ -2x & 2y \end{pmatrix}$$

The eigenvalues of the Jacobian at the equilibrium points are the following.

Case 1.

At $P_1(-1,-1)$ the Jacobian is $f'(-1,-1) = \begin{pmatrix} -2 & -8 \\ 2 & -2 \end{pmatrix}$

$\Longrightarrow \lambda_{1,2} = -2 \pm 4\,i$ (complex eigenvalues with negative real part)

$\Longrightarrow P_1(-1,-1)$ is a spiral sink

Case 2.

At $P_2(4,4)$ the Jacobian is $f'(4,4) = \begin{pmatrix} 8 & 2 \\ -8 & 8 \end{pmatrix}$

$\Longrightarrow \lambda_{1,2} = 8 \pm 4\,i$ (complex eigenvalues with positive real part)

$\Longrightarrow P_2(4,4)$ is a spiral source

The phase portrait of the system: