09 - Phase portraits for planar systems

Consider the two-dimensional linear differential equation system $X' = A X$, where $A \in \mathbb{R}^{2\times 2}$ and $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Let $Y = P X$ where $P \in \mathbb{R}^{2\times 2}$ is an invertible matrix (from here $X = P^{-1} Y$).

Then $Y' = P X' = P A X = P A P^{-1} Y$. So the differential equation system for the new variable is $Y' = B Y$ where $B = P A P^{-1}$.

Definition. The matrices A and B are **similar**, if there exists an invertible matrix P such that
$B = P A P^{-1}$.

Theorem. If A and B are similar then they have the same eigenvalues.

Consequence. It is enough to determine the phase portrait of the system $Y' = B Y$ since the phase portrait of the system $X' = A X$ can be obtained from this by the linear transformation $X = P^{-1} Y$.

Theorem. Any 2×2 matrix is similar to one of the following three matrices where $\lambda,\ \mu,\ \alpha,\ \beta \in \mathbb{R}$:

$$(1)\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad (2)\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad (3)\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

These matrices are in Jordan canonical form.

First suppose that in the system $X' = A X$, the coefficient matrix A is nonsingular (that is, $\det(A) \neq 0$, or zero is not an eigenvalue of A). In this case, $X' = 0 \implies X = 0$, that is, the origin is the only equilibrium point ($x(t) \equiv y(t) \equiv 0$).

Case (1): real eigenvalues, two eigenvectors

$$X' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} X \iff \begin{cases} x' = \lambda x, & x(0) = x_0 \\ y' = \mu y, & y(0) = y_0 \end{cases}$$

The solution is $\begin{cases} x(t) = x_0\, e^{\lambda t} \\ y(t) = y_0\, e^{\mu t} \end{cases}$ From here y can be expressed by x such that t is eliminated:

$$\begin{cases} x^{\mu} = {x_0}^{\mu} \cdot e^{\lambda \mu t} \\ y^{\lambda} = {y_0}^{\lambda} \cdot e^{\lambda \mu t} \end{cases} \implies \frac{x^{\mu}}{y^{\lambda}} = \frac{{x_0}^{\mu}}{{y_0}^{\lambda}} \implies y = \left(\frac{{y_0}^{\lambda}}{{x_0}^{\mu}}\right)^{1/\lambda} \cdot x^{\mu/\lambda} \implies y(x) = C \cdot x^{\mu/\lambda}$$

It means that the trajectories are graphs of power functions.

If λ and μ are of the same sign and if $\frac{\mu}{\lambda} > 1$ then the graphs are similar to a standing parabola.

If λ and μ are of the same sign and if $0 < \frac{\mu}{\lambda} < 1$ then the graphs are similar to a lying parabola.

If λ and μ are of the opposite sign then the graphs are similar to a hyperbola.

Observation: $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} x_0\, e^{\lambda t} = \begin{cases} +\infty, & \text{if } \lambda > 0,\ x_0 > 0 \\ -\infty, & \text{if } \lambda > 0,\ x_0 < 0 \\ 0, & \text{if } \lambda < 0 \end{cases}$

Examples.

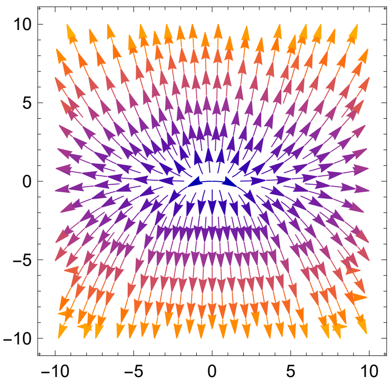
a) $\begin{cases} x' = 2x, & x(0) = x_0 \\ y' = 5y, & y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0\, e^{2t} \\ y = y_0\, e^{5t} \end{cases} \Longrightarrow \begin{cases} x^5 = {x_0}^5 \cdot e^{10t} \\ y^2 = {y_0}^2 \cdot e^{10t} \end{cases} \Longrightarrow \dfrac{x^5}{y^2} = \dfrac{{x_0}^5}{{y_0}^2} \Longrightarrow y(x) = C\, x^{\frac{5}{2}}$

The phase portrait in the case when $\lambda > 0,\ \mu > 0$: the trajectories tend away from the origin.

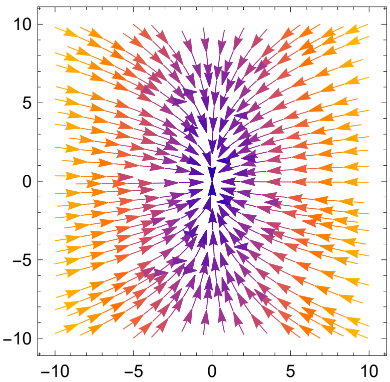
b) $\begin{cases} x' = -5x, & x(0) = x_0 \\ y' = -2y, & y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0\, e^{-5t} \\ y = y_0\, e^{-2t} \end{cases} \Longrightarrow \begin{cases} x^2 = {x_0}^2 \cdot e^{-10t} \\ y^5 = {y_0}^5 \cdot e^{-10t} \end{cases} \Longrightarrow \dfrac{x^2}{y^5} = \dfrac{{x_0}^2}{{y_0}^5} \Longrightarrow y(x) = C\, x^{\frac{2}{5}}$

The phase portrait in the case when $\lambda < 0,\ \mu < 0$: the trajectories tend towards the origin.

a)

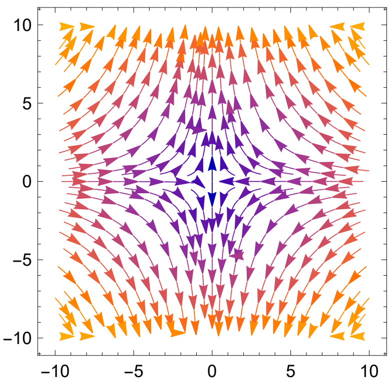


b)



c) $\begin{cases} x' = -2x, & x(0) = x_0 \\ y' = 3y, & y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0\, e^{-2t} \\ y = y_0\, e^{3t} \end{cases} \Longrightarrow \begin{cases} x^3 = {x_0}^3 \cdot e^{-6t} \\ y^2 = {y_0}^2 \cdot e^{6t} \end{cases} \Longrightarrow x^3 y^2 = {x_0}^3 {y_0}^2 \Longrightarrow y(x) = C\, x^{-\frac{3}{2}}$

The phase portrait in the case when $\lambda < 0,\ \mu > 0$: the trajectories on the x axis tend towards the origin and the trajectories on the y axis tend away from the origin.



Definition. Consider the system $X' = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} X$, where $\lambda\mu \neq 0$.

a) If $\lambda > 0,\ \mu > 0$ then the origin is a **source**.

b) If $\lambda < 0,\ \mu < 0$ then the origin is a **sink**.

c) If λ and μ are of the opposite sign then the origin is a **saddle**.

Other terms: real source or nodal source; real sink or nodal sink; node: sink or source.

Example. Sketch the phase portrait of the system $X' = AX$, where $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$.

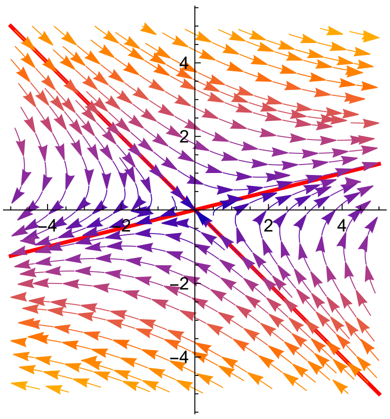
The eigenvalues and eigenvectors of A are $\lambda_1 = -3$, $\boldsymbol{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 2$, $\boldsymbol{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

The straight-line solutions are:

$X_1(t) = c_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \longrightarrow +\infty$.

$X_2(t) = c_2 e^{2t} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$: tends away from the origin as $t \longrightarrow +\infty$.

The general solution is $X(t) = X_1(t) + X_2(t)$. It can be seen that $X(t) \longrightarrow X_2(t)$ as $t \longrightarrow +\infty$ and $X(t) \longrightarrow X_1(t)$ as $t \longrightarrow -\infty$. The origin is a saddle.



Case (2): double real eigenvalues, one eigenvector only

$$X' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X \iff \begin{cases} x' = \lambda x + y, & x(0) = x_0 \\ y' = \lambda y, & y(0) = y_0 \end{cases}$$

From the second equation $y(t) = y_0 e^{\lambda t}$. Substituting into the first one, we obtain a first-order linear nonhomogeneous differential equation with constant coefficients, where there is outer resonance:

$$x' - \lambda x = y_0 e^{\lambda t}$$

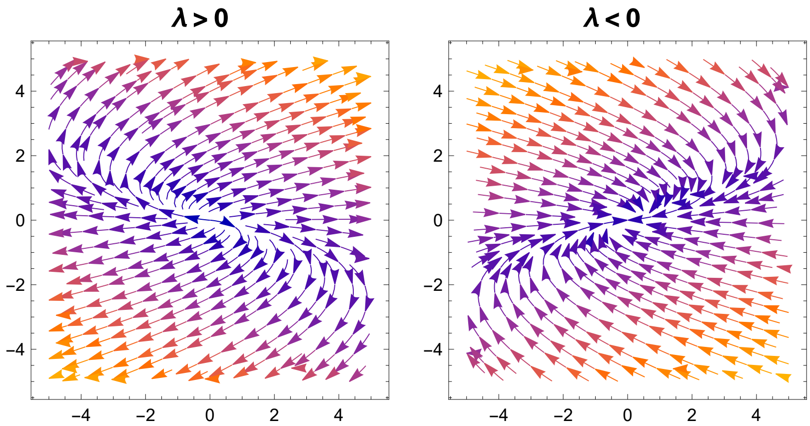
The general solution of the homogeneous equation is $x_h(t) = c e^{\lambda t}$.
The particular solution of the nonhomogeneous equation is $x_p(t) = A t e^{\lambda t} \Longrightarrow$

$$A e^{\lambda t} + \lambda A t e^{\lambda t} - \lambda A t e^{\lambda t} = y_0 e^{\lambda t} \Longrightarrow A = y_0$$

The general solution of the nonhomogeneous equation is $x(t) = x_h(t) + x_p(t) = c e^{\lambda t} + y_0 t e^{\lambda t}$.
From the initial condition: $x(0) = x_0 \Longrightarrow c = x_0$

The solution of the initial value problem is $x(t) = e^{\lambda t}(x_0 + y_0 t)$
$y(t) = y_0 e^{\lambda t}$

The phase portraits can be seen in the following figures. If $\lambda > 0$ then the trajectories tend away from the origin and if if $\lambda < 0$ then the trajectories tend towards the origin.



Definition. Consider the system $X' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} X$, where $\lambda \neq 0$.

a) If $\lambda > 0$ then the origin is a **degenerate source**.
b) If $\lambda < 0$ then the origin is a **degenerate sink**.

Other terms: improper node: improper sink/source.

Case (3): complex eigenvalues

$$X' = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} X \iff \begin{cases} (1)\, x' = \alpha x - \beta y \\ (2)\, y' = \beta x + \alpha y \end{cases}$$

In order to plot the solutions, we introduce polar coordinates:

$$x(t) = r(t) \cdot \cos(\varphi(t))$$
$$y(t) = r(t) \cdot \sin(\varphi(t))$$

Then from the equation system we obtain:

$$\begin{cases} (1)\, r' \cos\varphi - r\,\varphi' \sin\varphi = \alpha\, r \cos\varphi - \beta\, r \sin\varphi \\ (2)\, r' \sin\varphi + r\,\varphi' \cos\varphi = \beta\, r \cos\varphi + \alpha\, r \sin\varphi \end{cases}$$

$(1) \cdot \cos\varphi + (2) \cdot \sin\varphi \implies r' \left(\cos^2\varphi + \sin^2\varphi\right) = \alpha\, r \left(\cos^2\varphi + \sin^2\varphi\right)$, that is, $r' = \alpha\, r$
$(2) \cdot \cos\varphi - (1) \cdot \sin\varphi \implies r\,\varphi' \left(\cos^2\varphi + \sin^2\varphi\right) = \beta\, r \left(\cos^2\varphi + \sin^2\varphi\right)$, that is, $\varphi' = \beta$

Thus, the differential equation system in polar coordinates is the following:

$$\begin{cases} r'(t) = \alpha\, r(t), & r(0) = r_0 \\ \varphi'(t) = \beta, & \varphi(0) = \varphi_0 \end{cases}$$

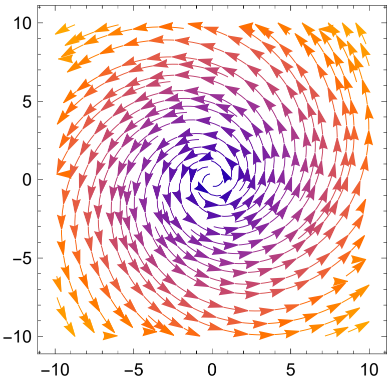
The solution of the initial value problem is:

$$\begin{cases} r(t) = r_0\, e^{\alpha t} \\ \varphi(t) = \beta\, t + \varphi_0 \end{cases}$$

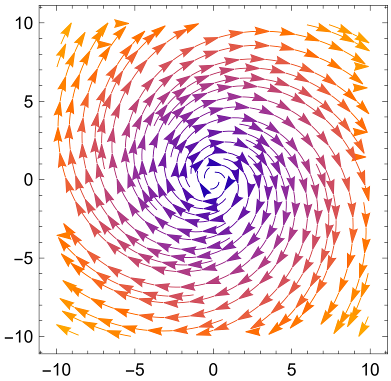
Consequence: The trajectories are such curves that go around the origin infinitely many times. The stability of the origin depends on the sign of α. (It depends on the sign of β whether the curves go in positive or negative direction around the origin.)

a) $\boldsymbol{\alpha > 0} \implies \lim\limits_{t\to\infty} r(t) = \lim\limits_{t\to\infty} r_0\, e^{\alpha t} = \infty \implies$ the trajectories tend away from the origin as t increases since $r(t)$ is an increasing function.

$x' = x - 3y$
$y' = 3x + y$

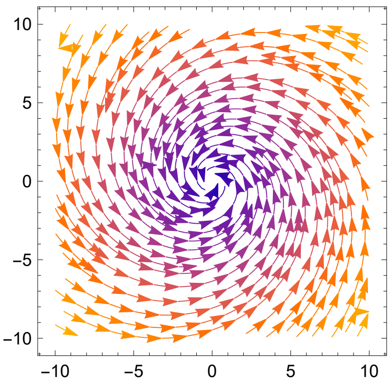


$x' = x + 3y$
$y' = -3x + y$

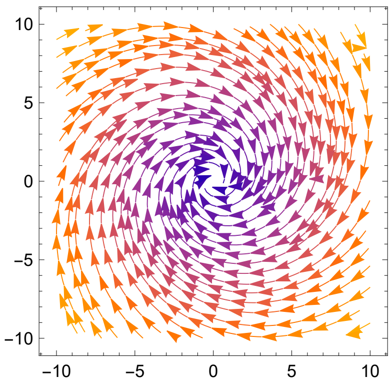


b) $\boldsymbol{\alpha < 0} \implies \lim\limits_{t\to\infty} r(t) = \lim\limits_{t\to\infty} r_0\, e^{\alpha t} = 0 \implies$ the trajectories tend towards the origin as t increases since $r(t)$ is a decreasing function.

$x' = -x - 3y$
$y' = 3x - y$

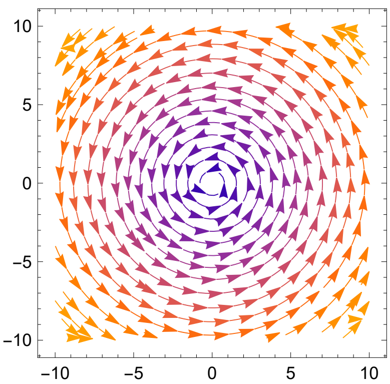


$x' = -x + 3y$
$y' = -3x - y$

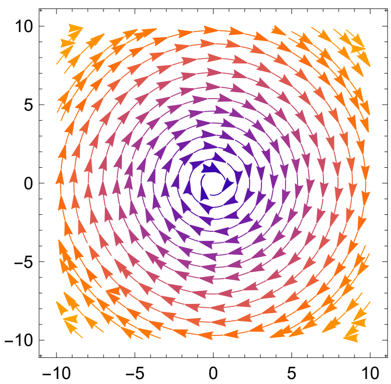


c) $\boldsymbol{\alpha = 0} \implies$ the trajectories are circles since $r(t) = r_0$ is a constant function.

$x' = -3y$
$y' = 3x$



$x' = 3y$
$y' = -3x$



Definition. Consider the system $X' = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} X$, where the eigenvalues of the coefficient matrix are $\lambda_{12} = \alpha \pm \beta i$.

a) If $\alpha > 0$ (the real part of the eigenvalues are positive) then the origin is a **spiral source**.

b) If $\alpha < 0$ (the real part of the eigenvalues are negative) then the origin is a **spiral sink**.

c) If $\alpha = 0$ then the origin is a **center**.

Other terms: focus: spiral sink or spiral source.

Case (4): Singular coefficient matrix

Now suppose that in the system $X' = AX$, the coefficient matrix A is **singular** (that is, zero is an eigenvalue of A). In this case the origin is not the only equilibrium point but also all points of a straight line passing through the origin (this is the line of the eigenvector that corresponds to the zero eigenvalue). It can also be the case that all points of the plane are equilibrium points.

There are four possible phase portraits (**degenerate cases**):

a) In case (1), $\lambda = 0$, $\mu > 0$, that is, $A = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} \Longleftrightarrow \begin{cases} x' = 0 \\ y' = \mu y \end{cases}$

$\Longrightarrow x(t) \equiv$ constant and $y(t) = y_0 e^{\mu t} \longrightarrow \begin{cases} \infty, & \text{if } y_0 > 0 \\ -\infty, & \text{if } y_0 < 0 \end{cases}$ as $t \longrightarrow \infty$.

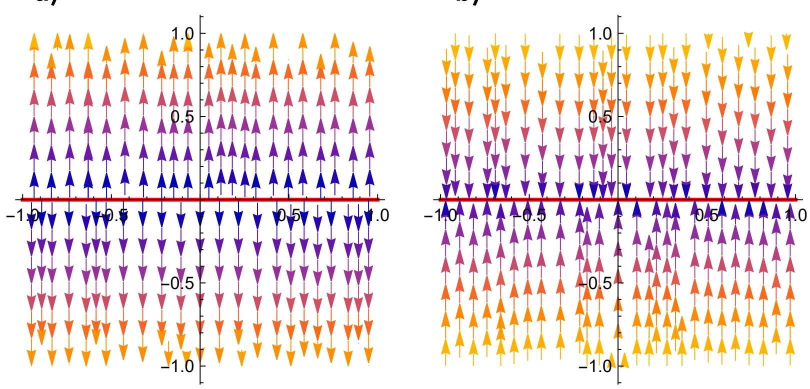
$\Longrightarrow$ All points of the x axis are equilibrium points. All the other trajectories are vertical half-lines that tend away from the x axis.

b) In case (1), $\lambda = 0$, $\mu < 0$, that is, $A = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} \Longleftrightarrow \begin{cases} x' = 0 \\ y' = \mu y \end{cases}$

$\Longrightarrow x(t) \equiv$ constant and $y(t) = y_0 e^{\mu t} \longrightarrow 0$ as $t \longrightarrow \infty$.

$\Longrightarrow$ All points of the x axis are equilibrium points. All the other trajectories are vertical half-lines that tend towards the x axis.

a) **b)**



c) In case (1), $\lambda = 0$, $\mu = 0$, that is, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff \begin{cases} x' = 0 \\ y' = 0 \end{cases}$

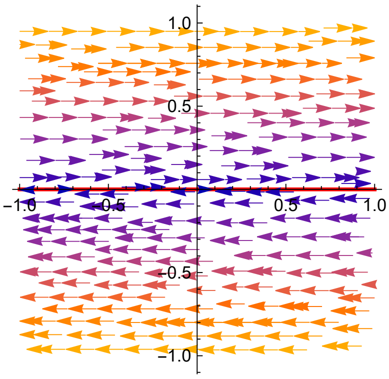
$\Longrightarrow x(t) \equiv$ constant and $y(t) \equiv$ constant

$\Longrightarrow$ All points of the plane are equilibrium points.

d) In case (2), $\lambda = 0$, that is, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \iff \begin{cases} x' = y \\ y' = 0 \end{cases}$

$\Longrightarrow y(t) \equiv c$ (constant) and $x'(t) = c \Longrightarrow \begin{cases} x \text{ increases if } c > 0 \\ x \text{ decreases if } c < 0 \end{cases}$

$\Longrightarrow$ All points of the x axis are equilibrium points. All the other trajectories are horizontal lines such that above the x axis the trajectories move in positive direction and below the x axis the trajectories move in negative direction.



The inverse transform

So far we have determined the phase portrait of the linear system $X' = AX$ where A is in Jordan canonical form. We have seen that with the substitution $Y = PX$, the above linear system can be transformed into $Y' = BY$ where $B = PAP^{-1}$.

Thus, the phase portrait of an arbitrary linear system can be obtained from the phase portrait corresponding to a Jordan canonical form by applying the inverse linear transformation P^{-1}.

During this linear transformation, in cases (1) and (2), the axes of the sink, source and saddle will be rotated and the axes will be identical to the straight lines of the eigenvectors of the coefficient matrix.

In case (3), the spirals around the origin may be distorted, and in the case of a center, we obtain ellipses instead of circles.

Finding the Jordan canonical form of a matrix

Example. Let $A = \begin{pmatrix} 5 & -3 \\ 2 & -2 \end{pmatrix}$. Find the matrix T that puts A in canonical form.

Remark: Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $\det P = a\,d - b\,c \neq 0$. Then the inverse of P is $P^{-1} = \dfrac{1}{a\,d - b\,c} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Solution: The eigenvalues and eigenvectors of A are $\lambda_1 = 4$, $u = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\lambda_2 = -1$, $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Let T be the matrix whose columns are the eigenvectors: $T = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.

The inverse of T is $T^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$. So

$B = T^{-1} A\, T = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 12 & -1 \\ 4 & -2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.