09 - Phase portraits for planar systems, solutions

Exercise 1.

a) $x' = -2x + 2y, \quad y' = x - y$

$A = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}$

The eigenvalues and eigenvectors of A: $\lambda_1 = -3, \quad \boldsymbol{u} = (-2, 1)$

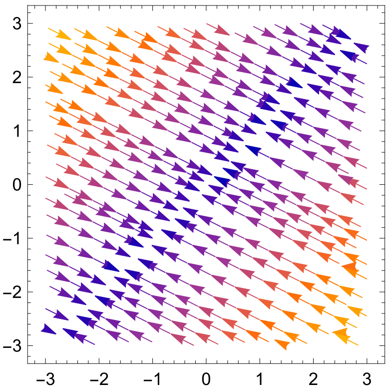
$\lambda_2 = 0, \quad \boldsymbol{v} = (1, 1)$

The coefficient matrix is singular (one eigenvalue is zero), so this is a degenerate case.

All points of a straight line passing through the origin are singular points, this is the straight line of the eigenvector $\boldsymbol{v} = (1, 1)$, since the corresponding eigenvalue is $\lambda_2 = 0$.

The trajectories are half-lines that are parallel to the eigenvector $\boldsymbol{u} = (-2, 1)$.

Since the corresponding eigenvalue is $\lambda_1 = -3 < 0$ then the trajectories tend towards the straight line of $\boldsymbol{v} = (1, 1)$.



b) $x' = -x + 2y, \quad y' = x + y$

$A = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$

1) The eigenvalues and eigenvectors of A: $\lambda_1 = -\sqrt{3}, \quad u = \left(-1 - \sqrt{3}, 1\right)$

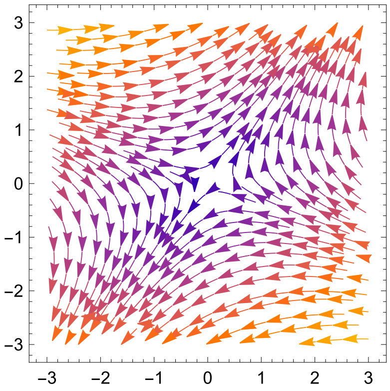
$\lambda_2 = \sqrt{3}, \quad v = \left(-1 + \sqrt{3}, 1\right)$

2) $\text{trace}(A) = T = -1 + 1 = 0$

$\det(A) = D = (-1) \cdot 1 - 2 \cdot 1 = -3 < 0$

$T^2 - 4D = 0 - 4(-3) = 12 > 0$

$\Longrightarrow$ positive and negative real eigenvalues $\Longrightarrow$ the origin is a saddle



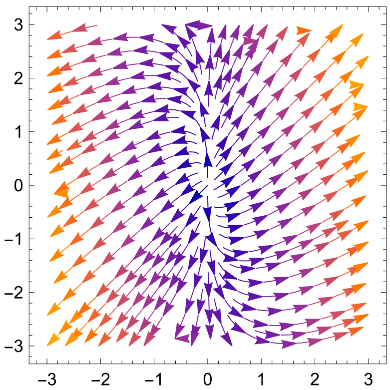
c) $x' = 3x, \quad y' = 2x + y$

$A = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$

1) The eigenvalues and eigenvectors of A: $\lambda_1 = 3, \quad u = (1, 1)$
$\lambda_2 = 1, \quad v = (0, 1)$

2) $\text{trace}(A) = T = 3 + 1 = 4 > 0$
$\det(A) = D = 3 \cdot 1 - 0 \cdot 2 = 3 > 0$
$T^2 - 4D = 16 - 12 = 4 > 0$

$\Longrightarrow$ two different positive real eigenvalues $\Longrightarrow$ the origin is a (real) source



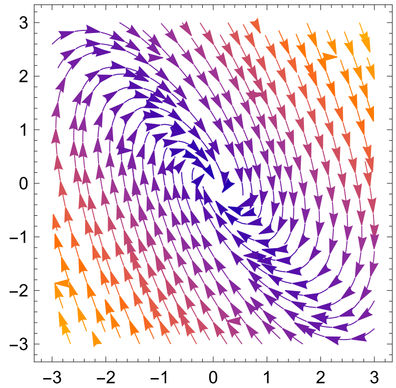
d) $x' = x + 3y, \quad y' = -6x - 5y$

$A = \begin{pmatrix} 1 & 3 \\ -6 & -5 \end{pmatrix}$

1) The eigenvalues of A: $\lambda_1 = -2 + 3i$
$\lambda_2 = -2 - 3i$

2) $\text{trace}(A) = T = 1 - 5 = -4 < 0$
$\det(A) = D = -5 - (-18) = 13$
$T^2 - 4D = 16 - 4 \cdot 13 = -36 < 0$

$\Longrightarrow$ complex eigenvalues with negative real part $\Longrightarrow$ the origin is a spiral sink



e) $x' = -4x + \frac{1}{2}y, \quad y' = 2x - 4y$

$$A = \begin{pmatrix} -4 & \frac{1}{2} \\ 2 & -4 \end{pmatrix}$$

1) The eigenvalues and eigenvectors of A: $\lambda_1 = -5, \quad u = \left(-\frac{1}{2}, 1\right)$

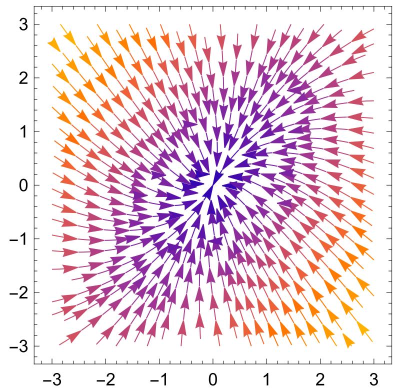
$\lambda_2 = -3, \quad v = \left(\frac{1}{2}, 1\right)$

2) $\text{trace}(A) = T = -4 - 4 = -8 < 0$

$\det(A) = D = 16 - 1 = 15 > 0$

$T^2 - 4D = 64 - 60 = 4 > 0$

$\Longrightarrow$ two different negative real eigenvalues $\Longrightarrow$ the origin is a (real) sink



f) $x' = -2x - 5y, \quad y' = 2x + 2y$

$$A = \begin{pmatrix} -2 & -5 \\ 2 & 2 \end{pmatrix}$$

1) The eigenvalues of A: $\lambda_1 = i\sqrt{6}$

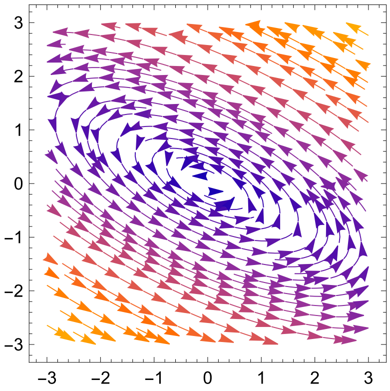
$\lambda_2 = -i\sqrt{6}$

2) $\text{trace}(A) = T = -2 + 2 = 0$

$\det(A) = D = -4 - (-10) = 6$

$T^2 - 4D = 0 - 24 = -24 < 0$

$\Longrightarrow$ complex eigenvalues with zero real part $\Longrightarrow$ the origin is a center



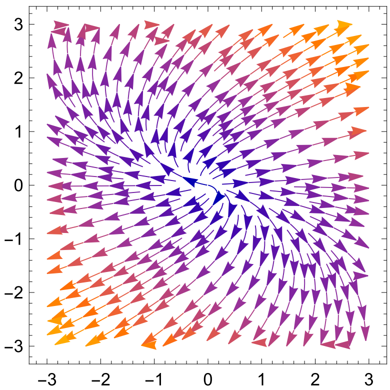
g) $x' = x + y, \quad y' = y$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues and eigenvectors of A: $\lambda_1 = \lambda_2 = 1, \quad u = (1, 0)$

$\Longrightarrow$ double real positive eigenvalue, one eigenvector only

$\Longrightarrow$ the origin is a degenerate source



h) $x' = x + 2y, \quad y' = -2x + y$

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

1) The eigenvalues of A: $\lambda_1 = 1 + 2i$

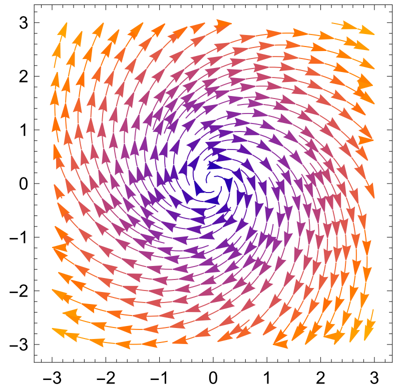
$\lambda_2 = 1 - 2i$

2) $\text{trace}(A) = T = 1 + 1 = 2 > 0$

$\det(A) = D = 1 - (-4) = 5 > 0$

$T^2 - 4D = 4 - 4 \cdot 5 = -16 < 0$

$\Longrightarrow$ complex eigenvalues with positive real part $\Longrightarrow$ the origin is a spiral source



Exercise 2.

2. a) first solution

The coefficient matrix is $A = \begin{pmatrix} -3 & a \\ 2 & 1 \end{pmatrix}$.

The characteristic equation: $\det\begin{pmatrix} -3-\lambda & a \\ 2 & 1-\lambda \end{pmatrix} = (-3-\lambda)(1-\lambda) - 2a =$

$\lambda^2 + 2\lambda - 3 - 2a = 0$

The eigenvalues of A are $\lambda_{1,2} = \dfrac{-2 \pm \sqrt{4 - 4(-3-2a)}}{2} = -1 \pm \sqrt{4+2a}$

1) If $4 + 2a < 0$, that is, $a < -2$, then A has complex eigenvalues with negative real part
$\Longrightarrow$ the origin is a spiral sink

2) If $a = -2$, then A has repeated eigenvalues $-1 \Longrightarrow$ the origin is a degenerate sink

3) If $a > -2$, then A has real distinct eigenvalues: $\lambda_1 = -1 + \sqrt{4+2a}$ and $\lambda_2 = -1 - \sqrt{4+2a}$.

It can be seen that λ_2 is always negative but λ_1 can be positive, zero or negative.

a) If $-1 + \sqrt{4+2a} = 0$, that is, $a = -1.5$, then $\lambda_1 = 0$, $\lambda_2 = -2$
$\Longrightarrow$ it is a degenerate case, the phase portrait consists of parallel straight lines

b) If $-1 + \sqrt{4+2a} < 0$, that is, $-2 < a < -1.5$, then $\lambda_1 < 0$, $\lambda_2 < 0 \Longrightarrow$ the origin is a sink

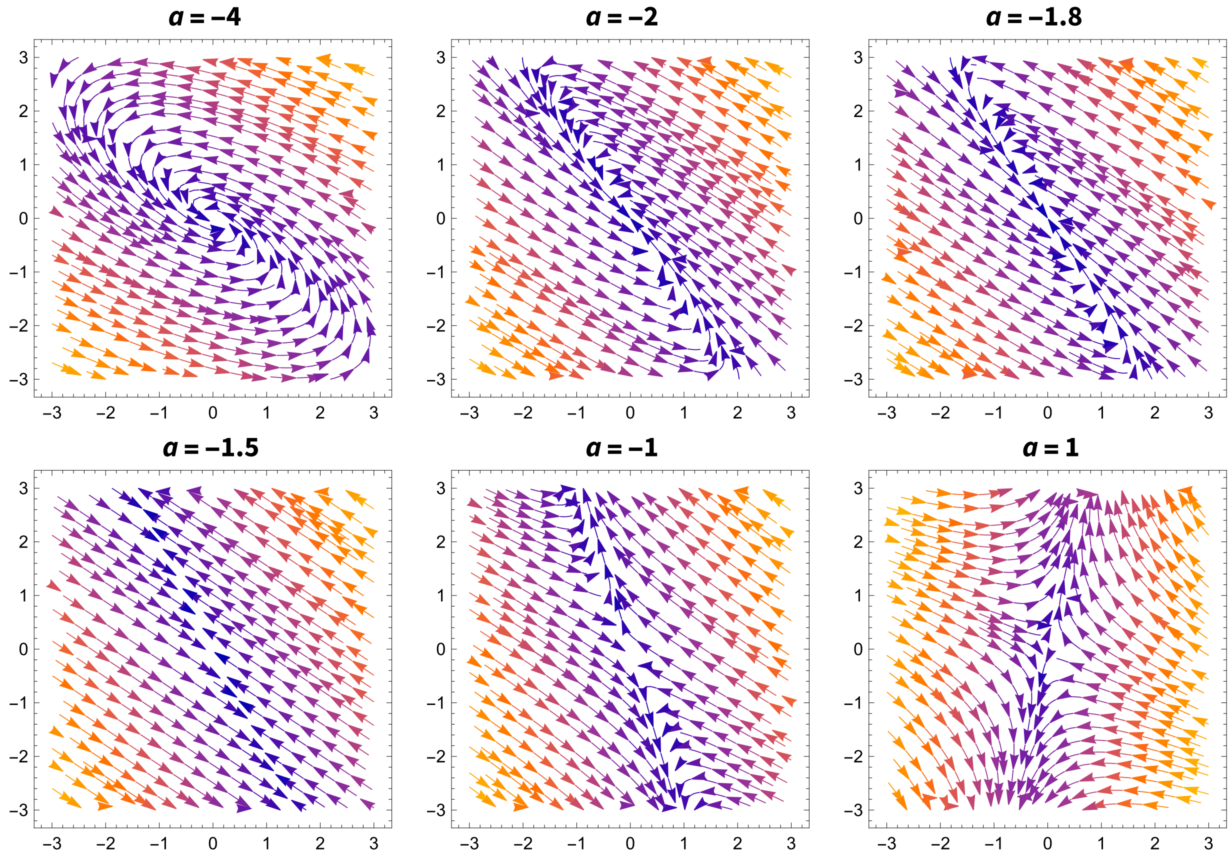
c) If $-1 + \sqrt{4+2a} > 0$, that is, $a > -1.5$, then $\lambda_1 > 0$, $\lambda_2 < 0 \Longrightarrow$ the origin is a saddle

The type of the origin:
$a < -2$: spiral sink

$-2 < a < -1.5$: (real) sink

$a > -1.5$: saddle

The system has a bifurcation at $a = -2$ and $a = -1.5$.
A few phase portraits can be seen in the following figures.



2. a) second solution

The trace and determinant of $A = \begin{pmatrix} -3 & a \\ 2 & 1 \end{pmatrix}$ are: $T = -3 + 1 = -2$

$D = -3 - 2a$

$T^2 - 4D = 4 - 4(-3 - 2a) = 16 + 8a$

We investigate the following six non-degenerate cases:

1. $T^2 - 4D < 0,\ T < 0 \iff 16 + 8a < 0 \iff a < -2 \implies$ the origin is a spiral sink
2. $T^2 - 4D < 0,\ T > 0$: cannot be the case
3. $T^2 - 4D < 0,\ T = 0$: cannot be the case
4. $T^2 - 4D > 0,\ D < 0 \iff 16 + 8a > 0$ and $-3 - 2a < 0 \iff a > -2$ and $a > -1.5$
   $\iff a > -1.5 \implies$ the origin is a saddle
5. $T^2 - 4D > 0,\ D > 0,\ T > 0$: cannot be the case
6. $T^2 - 4D > 0,\ D > 0,\ T < 0 \iff 16 + 8a > 0$ and $-3 - 2a > 0 \iff -2 < a < -1.5 \implies$ the origin is a sink

The type of the origin:

$a < -2$: spiral sink

$-2 < a < -1.5$: (real) sink

$a > -1.5$: saddle

The system has a bifurcation at $a = -2$ and $a = -1.5$ (these are degenerate cases).

2. b)

The characteristic equation of $A = \begin{pmatrix} 3 & 1 \\ p & -1 \end{pmatrix}$ is $(3-\lambda)(-1-\lambda) - p = \lambda^2 - 2\lambda - (3+p) = 0$

The eigenvalues of A are $\lambda_{1,2} = \dfrac{2 \pm \sqrt{4 + 4(3+p)}}{2} = \dfrac{2 \pm \sqrt{16 + 4p}}{2} = 1 \pm \sqrt{4+p}$

1) If $p < -4$, then A has complex eigenvalues with positive real part $\Longrightarrow$ the origin is a spiral source

2) If $p = -4$, then both eigenvalues are equal to 1 $\Longrightarrow$ the origin is a degenerate source

3) If $p > -4$, then A has real distinct eigenvalues: $\lambda_1 = 1 + \sqrt{4+p}$ and $\lambda_2 = 1 - \sqrt{4+p}$.

It can be seen that λ_1 is always positive but λ_2 can be positive, zero or negative.

a) If $1 - \sqrt{4+p} = 0$, that is, $p = -3$, then $\lambda_1 = 2$, $\lambda_2 = 0$

$\Longrightarrow$ it is a degenerate case, the phase portrait consists of parallel straight lines

b) If $1 - \sqrt{4+p} > 0 \Longrightarrow \sqrt{4+p} < 1 \Longrightarrow -4 < p < -3$, then $\lambda_1 > 0$, $\lambda_2 > 0$ $\Longrightarrow$ the origin is a source

c) If $1 - \sqrt{4+p} < 0 \Longrightarrow \sqrt{4+p} > 1 \Longrightarrow p > -3$, then $\lambda_1 > 0$, $\lambda_2 < 0$ $\Longrightarrow$ the origin is a saddle

The type of the origin:

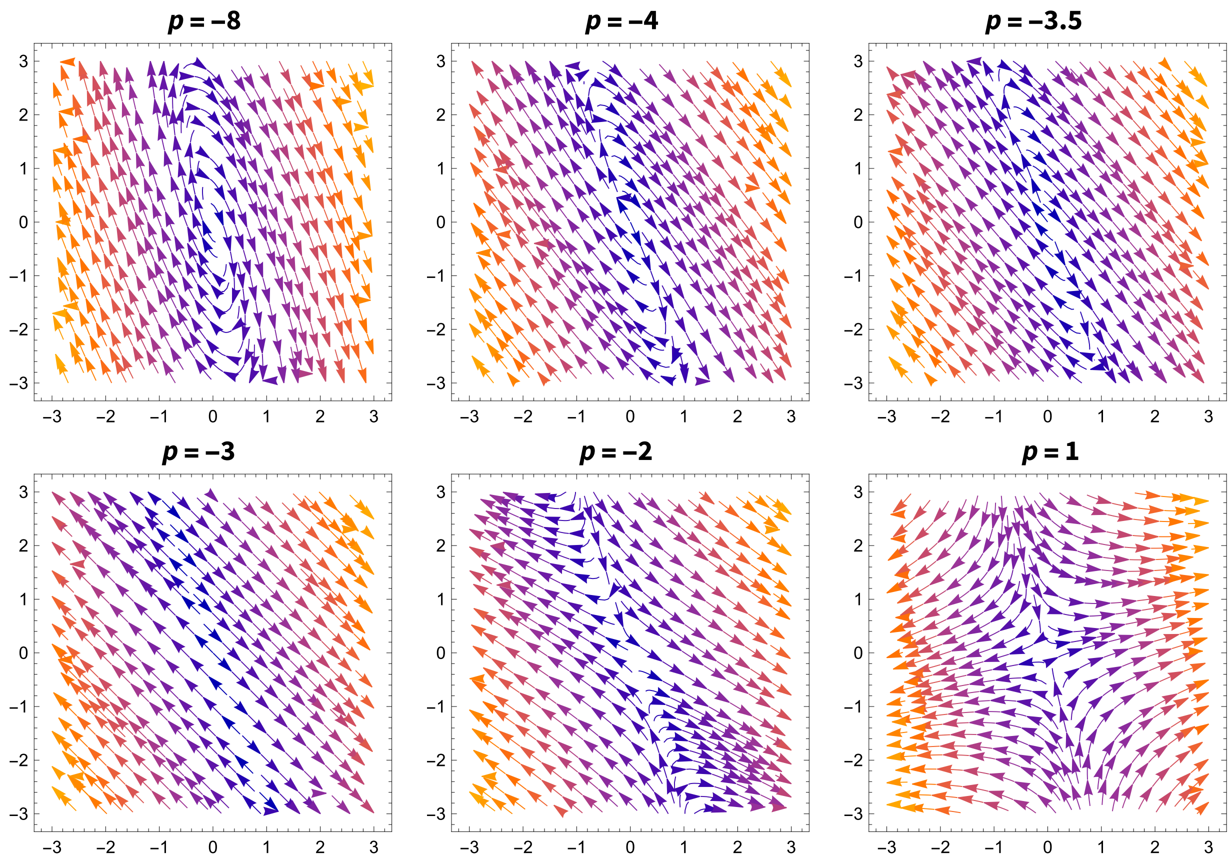
$p < -4$: spiral source

$-4 < p < -3$: source

$p > -3$: saddle

The system has a bifurcation at $p = -4$ and $p = -3$.

A few phase portraits can be seen in the following figures.



2. c)

The characteristic equation of $A = \begin{pmatrix} 2p & 3+p \\ 3-p & 0 \end{pmatrix}$ is

$\det\begin{pmatrix} 2p-\lambda & 3+p \\ 3-p & -\lambda \end{pmatrix} = (2p-\lambda)(-\lambda) - (3+p)(3-p) = \lambda^2 - 2p\lambda + (p^2-9) = 0$

The eigenvalues of A are $\lambda_{1,2} = \dfrac{2p \pm \sqrt{4p^2 - 4(p^2-9)}}{2} = p \pm 3$

It can be seen that $\lambda_1 = p-3$ and $\lambda_2 = p+3$ are always real.

Degenerate cases:

If $p = -3$, then $\lambda_1 = -6$, $\lambda_2 = 0$ $\Longrightarrow$ the phase portrait consists of parallel straight lines

If $p = 3$, then $\lambda_1 = 0$, $\lambda_2 = 6$ $\Longrightarrow$ the phase portrait consists of parallel straight lines

Non-degenerate cases:

1. $\lambda_1 > 0$ and $\lambda_2 > 0$ $\Longleftrightarrow$ $p > 3$ and $p > -3$ $\Longleftrightarrow$ $p > 3$ $\Longrightarrow$ the origin is a source
2. λ_1 and λ_2 are of opposite sign if and only if $-3 < p < 3$ $\Longrightarrow$ the origin is a saddle

   calculations: (i) $p-3 > 0$ and $p+3 < 0$ $\Longleftrightarrow$ $p > 3$ and $p < -3$: this cannot be the case

   (ii) $p-3 < 0$ and $p+3 > 0$ $\Longleftrightarrow$ $p < 3$ and $p > -3$
3. $\lambda_1 < 0$ and $\lambda_2 < 0$ if and only if $p < -3$ $\Longrightarrow$ the origin is a sink

The type of the origin: $p < -3$: sink, $-3 < p < 3$: saddle, $p > 3$: source

The system has a bifurcation at $p = -3$ and $p = 3$.

A few phase portraits can be seen in the following figures.