09 - Phase portraits for planar systems

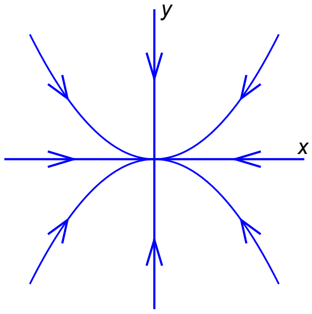
Summary

Consider the two-dimensional linear differential equation system $X' = AX$, where $A \in \mathbb{R}^{2\times 2}$ and $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Assume that $\det A \neq 0$ and the eigenvalues of A are $\lambda_1,\ \lambda_2$.

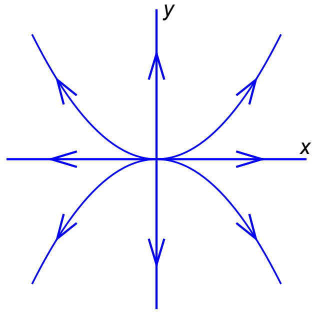
There are 6 non-degenerate cases.

Assume that $\lambda_1,\ \lambda_2 \in \mathbb{R}$ and A has two eigenvectors.

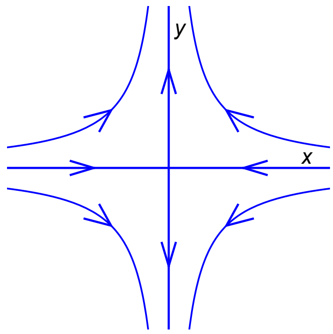
a) If $\lambda_1 < 0,\ \lambda_2 < 0 \Longrightarrow$
the origin is a sink



b) If $\lambda_1 > 0,\ \lambda_2 > 0 \Longrightarrow$
the origin is a source

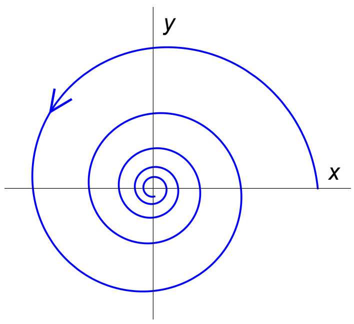


c) If $\lambda_1 < 0,\ \lambda_2 > 0$ (or $\lambda_1 > 0,\ \lambda_2 < 0$)
$\Longrightarrow$ the origin is a saddle

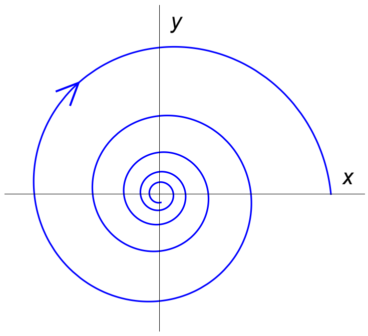


Assume that $\lambda_{1,2} = \alpha \pm \beta i$, where $\alpha,\ \beta \in \mathbb{R}$.

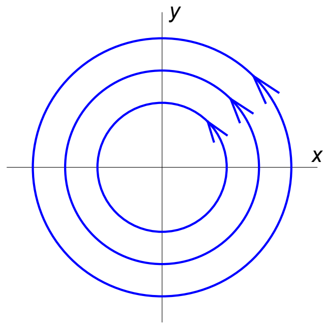
d) If $\text{Re}(\lambda_{1,2}) = \alpha < 0 \Longrightarrow$
the origin is a spiral sink



e) If $\text{Re}(\lambda_{1,2}) = \alpha > 0 \Longrightarrow$
the origin is a spiral source



f) f $\text{Re}(\lambda_{1,2}) = \alpha = 0 \Longrightarrow$
the origin is a center



The Trace-Determinant Plane

Determine the phase portrait of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The characteristic equation for A is

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - b\,c = \lambda^2 - (a+d)\,\lambda + (a\,d - b\,c) = 0.$$

The trace and determinant of A are

$$T := \mathrm{Tr}(A) = a + d$$
$$D := \det(A) = a\,d - b\,c$$

The eigenvalues are: $\lambda_{1,2} = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4\,(a\,d - b\,c)}}{2} = \dfrac{1}{2}\left(T \pm \sqrt{T^2 - 4\,D}\right)$.

Suppose that $T^2 - 4\,D \neq 0$. Then the type of the origin is the following.

1. $T^2 - 4\,D < 0,\; T < 0 \implies$ spiral sink

(complex eigenvalues with negative real part: $\lambda_{1,2} = \alpha \pm \beta\,i,\;\; \alpha < 0,\;\; \beta \neq 0$)

2. $T^2 - 4\,D < 0,\; T > 0 \implies$ spiral source

(complex eigenvalues with positive real part: $\lambda_{1,2} = \alpha \pm \beta\,i,\;\; \alpha > 0,\;\; \beta \neq 0$)

3. $T^2 - 4\,D < 0,\; T = 0 \implies$ center

(complex eigenvalues with zero real part: $\lambda_{1,2} = \pm\beta\,i,\;\; \beta \neq 0$)

4. $T^2 - 4\,D > 0,\; D < 0 \implies$ saddle

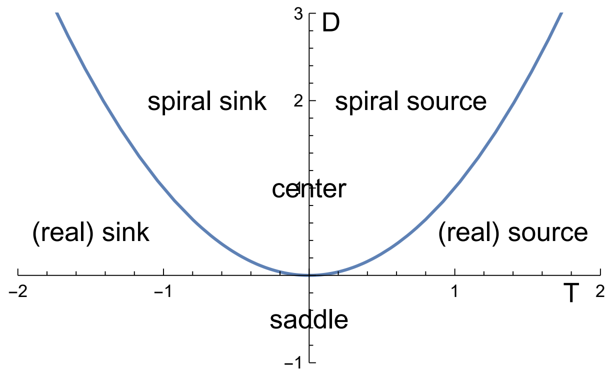
(positive and negative real eigenvalues: $\lambda_1 > 0,\;\; \lambda_2 < 0$)

5. $T^2 - 4\,D > 0,\; D > 0,\; T > 0 \implies$ (real) source

(two different positive real eigenvalues: $\lambda_1 > 0,\;\; \lambda_2 > 0$)

6. $T^2 - 4\,D > 0,\; D > 0,\; T < 0 \implies$ (real) sink

(two different negative real eigenvalues: $\lambda_1 < 0,\;\; \lambda_2 < 0$)



Exercises

1.

Sketch the phase portraits of the following systems. Solve the exercises in two ways, that is, using the eigenvalues and then using the trace and determinant of the coefficient matrix.

a) $x' = -2x + 2y, \quad y' = x - y$

b) $x' = -x + 2y, \quad y' = x + y$

c) $x' = 3x, \quad y' = 2x + y$

d) $x' = x + 3y, \quad y' = -6x - 5y$

e) $x' = -4x + \frac{1}{2}y, \quad y' = 2x - 4y$

f) $x' = -2x - 5y, \quad y' = 2x + 2y$

g) $x' = x + y, \quad y' = y$

h) $x' = x + 2y, \quad y' = -2x + y$

2.

Find the type of the following linear systems for those parameters for which the system has one equilibrium point. For which values of the parameter do you find a **bifurcation**?

a) $A = \begin{pmatrix} -3 & a \\ 2 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & 1 \\ p & -1 \end{pmatrix}$ c) $A = \begin{pmatrix} 2p & 3+p \\ 3-p & 0 \end{pmatrix}$

d) $A = \begin{pmatrix} 0 & 1+p \\ -1 & p \end{pmatrix}$ e) $A = \begin{pmatrix} p & -1 \\ 1 & 1 \end{pmatrix}$