08 - Linear differential equation systems

Introduction - Some linear algebra

Linear systems

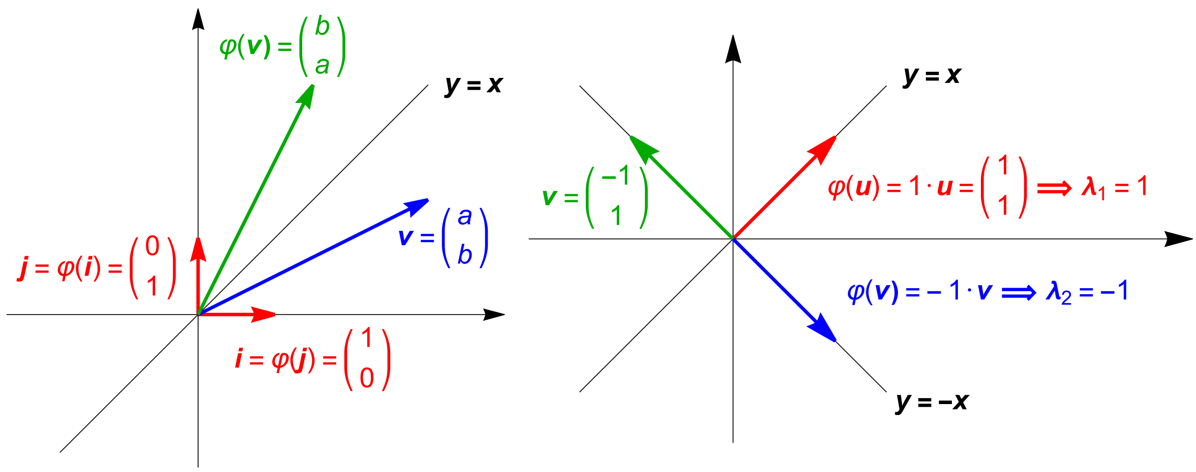
Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$ where $A \in \mathbb{R}^{n\times n}$, $\boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^n$.
This system has a unique solution if and only if $\det A \neq 0$.
If $\det A = 0$ then either there are infinitely many solutions or there are no solutions.

If $\boldsymbol{b} = \boldsymbol{0}$ then the linear system is homogeneous: $A\boldsymbol{x} = \boldsymbol{0}$. In this special case a solution always exists, it is the trivial solution $\boldsymbol{x} = \boldsymbol{0}$. Therefore, if $\boldsymbol{b} = \boldsymbol{0}$ and $\det A = 0$ then there are always infinitely many solutions.

Eigenvalue and eigenvector

Definition: Let $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{v} \neq \boldsymbol{0}$. We say that $\boldsymbol{v}$ is an eigenvector of the matrix A with eigenvalue λ if $A\boldsymbol{v} = \lambda\boldsymbol{v}$.

Example. Let the linear transformation $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the reflection about the straight line $y = x$. The matrix of φ in the basis $\boldsymbol{i}, \boldsymbol{j}$ is the matrix whose columns are the images of $\boldsymbol{i}, \boldsymbol{j}$, respectively: $A = (\varphi(\boldsymbol{i}) \mid \varphi(\boldsymbol{j})) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the image of an arbitrary vector $\boldsymbol{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is $\varphi(\boldsymbol{v}) = A\boldsymbol{v} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$



It can be seen that the eigenvalues and eigenvectors of this linear transformation are $\lambda_1 = 1$, $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -1$, $\boldsymbol{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The geometrical meaning of this is that the images of $\boldsymbol{u}$ and $\boldsymbol{v}$ are $\varphi(\boldsymbol{u}) = 1 \cdot \boldsymbol{u}$ and $\varphi(\boldsymbol{v}) = -1 \cdot \boldsymbol{v}$.

Remark 1: It is obvious that if $\boldsymbol{v}$ is an eigenvector of A and $\alpha \neq 0$ then $\alpha\, \boldsymbol{v}$ is also an eigenvector of A.

Remark 2: Let $I_n \in \mathbb{R}^{n \times n}$ be the identity matrix. Then $A\, \boldsymbol{v} = \lambda\, \boldsymbol{v} \iff A\, \boldsymbol{v} - \lambda\, I_n\, \boldsymbol{v} = (A - \lambda\, I_n)\, \boldsymbol{v} = \boldsymbol{0}$.
This is a homogeneous linear system for the coordinates of the eigenvector $\boldsymbol{v}$, where $\boldsymbol{v} \neq \boldsymbol{0}$. Thus, the equation holds if and only if $\det(A - \lambda\, I_n) = 0$.

$\Longrightarrow$ The eigenvalues of the matrix can be determined from the **characteristic equation** $\det(A - \lambda\, I_n) = 0$ and then the eigenvectors from the definition $A\, \boldsymbol{v} = \lambda\, \boldsymbol{v}$.

Examples

Exercise. Calculate the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.

Solution. The eigenvalues of A can be determined from the characteristic equation:

$$\det(A - \lambda\, I_2) = \det\left(\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{pmatrix} =$$

$$= (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\,\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \implies \lambda_1 = 4,\ \ \lambda_2 = -1$$

The eigenvectors of A can be determined from the definition:

- If $\lambda_1 = 4$ and $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ then $A\, \boldsymbol{u} = \lambda_1\, \boldsymbol{u} \iff \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 4\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \iff$

$$\begin{matrix} u_1 + 3\,u_2 = 4\,u_1 \\ 2\,u_1 + 2\,u_2 = 4\,u_2 \end{matrix} \iff \begin{matrix} 3\,u_2 = 3\,u_1 \\ 2\,u_1 = 2\,u_2 \end{matrix} \iff u_1 = u_2$$

$\implies$ For example, $u_1 = u_2 = 1$ is a suitable choice, so $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 4$.

- If $\lambda_2 = -1$ and $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $A\, \boldsymbol{v} = \lambda_2\, \boldsymbol{v} \iff \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -1 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff$

$$\begin{matrix} v_1 + 3\,v_2 = -v_1 \\ 2\,v_1 + 2\,v_2 = -v_2 \end{matrix} \iff \begin{matrix} 3\,v_2 = -2\,v_1 \\ 2\,v_1 = -3\,v_2 \end{matrix} \iff 2\,v_1 = -3\,v_2$$

$\implies$ For example, $v_1 = -3$, $v_2 = 2$ is a suitable choice, so $\boldsymbol{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = -1$.

Another possible choice is for example: $v_1 = 10 \implies v_2 = -\frac{20}{3} \implies \boldsymbol{v} = \begin{pmatrix} 10 \\ -\frac{20}{3} \end{pmatrix}$.

Homework: Calculate the eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the above example.

First-order linear differential equation systems

Example: $x_1'(t) = 2x_1(t) + 3x_2(t) + 1$
$x_2'(t) = -x_1(t) + x_2(t) + e^t$

In a matrix form: $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ e^t \end{pmatrix}$ or

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t) \text{ where } \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}, \boldsymbol{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$$

This is a first-order (two-variable) differential equation system where $x_1(t)$, $x_2(t)$ are the unknown functions and t is the variable.

Definition. First-order non-homogeneous differential equation system with constant coefficients:

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}(t), \text{ where } \boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}, A \in \mathbb{R}^{n\times n}, \boldsymbol{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \dots \\ f_n(t) \end{pmatrix} \not\equiv \boldsymbol{0}$$

Homogeneous differential equation system:

$$\boldsymbol{x}' = A\boldsymbol{x}$$

Theorem: $\boldsymbol{x}(t) = \boldsymbol{x_h}(t) + \boldsymbol{x_p}(t)$

The general solution of the nonhomogeneous equation is equal to the sum of the general solution of the homogeneous equation and the particular solution of the nonhomogeneous equation.

Remark: In many cases, $\boldsymbol{x_p}(t)$ can be determined by the method of undetermined coefficients or by the method of variation of the constants.

Theorem: The solutions of the homogeneous equation $\boldsymbol{x}' = A\boldsymbol{x}$ (where $A \in \mathbb{R}^{n\times n}$) generate a linear space with dimension n.

Theorem: If λ is an eigenvalue of A and $\boldsymbol{v}$ is the corresponding eigenvector (that is, $A\boldsymbol{v} = \lambda\boldsymbol{v}$ where $\boldsymbol{v} \neq \boldsymbol{0}$) then the function $\boldsymbol{x}(t) = e^{\lambda t}\boldsymbol{v}$ is a solution of the homogeneous equation $\boldsymbol{x}' = A\boldsymbol{x}$.

Proof:

$$\boldsymbol{x}'(t) = \frac{d}{dt} \begin{pmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \\ \dots \\ e^{\lambda t} v_n \end{pmatrix} = \begin{pmatrix} \lambda e^{\lambda t} v_1 \\ \lambda e^{\lambda t} v_2 \\ \dots \\ \lambda e^{\lambda t} v_n \end{pmatrix} = \lambda e^{\lambda t}\boldsymbol{v} = e^{\lambda t}\lambda\boldsymbol{v} = e^{\lambda t} A\boldsymbol{v} = A e^{\lambda t}\boldsymbol{v} = A\boldsymbol{x}(t).$$

Theorem: Consider the homogeneous equation $\boldsymbol{x}' = A\boldsymbol{x}$ where $A \in \mathbb{R}^{n\times n}$.
If the coefficient matrix A has n different eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the corresponding eigenvectors are $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$, then the general solution of the homogeneous equation is

$$\boldsymbol{x}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2 + \ldots + c_n e^{\lambda_n t}\boldsymbol{v}_n$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Remark: The statement is true over the field of the real and complex numbers.

Solution method for planar linear differential equation systems

Consider the system $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$, where $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $A \in \mathbb{R}^{2\times 2}$.

The solution $\boldsymbol{x}(t)$ is given by the formula

$\boldsymbol{x}(t) = P e^{Bt}\boldsymbol{c}$ ($\boldsymbol{c} \in \mathbb{R}^2$ is arbitrary).

Let $P = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and let the eigenvalues of A be λ_1 and λ_2.

Case 1) If $\lambda_1 \neq \lambda_2$ are real eigenvalues then $\boldsymbol{u}$ and $\boldsymbol{v}$ are eigenvectors corresponding to λ_1 and λ_2, respectively, and $e^{Bt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$.

Case 2) If $\lambda_1 = \lambda_2 =: \lambda$ is a real eigenvalue then $\boldsymbol{u}$ is an eigenvector corresponding to λ and $A\boldsymbol{v} = \lambda\boldsymbol{v} + \boldsymbol{u}$, and $e^{Bt} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

Case 3) If $\lambda_{1,2} = \alpha \pm \beta i$ are complex eigenvalues ($\beta \neq 0$) then $\boldsymbol{u} = \mathrm{Re}(\boldsymbol{p})$ and $\boldsymbol{v} = \mathrm{Im}(\boldsymbol{p})$ where $\boldsymbol{p}$ is a complex eigenvector corresponding to λ_1 (for example) and
$e^{Bt} = e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$.

Exercises

Solve the following initial value problems.

1. $x_1' = 2x_1 + x_2 \qquad x_1(0) = 3$
   $x_2' = 3x_1 + 4x_2 \qquad x_2(0) = 1$

Solution. The coefficient matrix is $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$.

- The eigenvalues of A:

$$\det(A - \lambda I_2) = \det\begin{pmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5) = 0$$

$$\Longrightarrow \lambda_1 = 1, \ \lambda_2 = 5$$

A has distinct real eigenvalues, this is case 1) above.

- The eigenvectors of A:

If $\lambda_1 = 1$ and $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ then $A\,\boldsymbol{u} = \lambda_1\,\boldsymbol{u} \iff \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 1 \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \iff$

$$\begin{matrix} 2u_1 + u_2 = u_1 \\ 3u_1 + 4u_2 = u_2 \end{matrix} \iff u_1 = -u_2 \Longrightarrow u_2 = 1, \ u_1 = -1 \Longrightarrow \boldsymbol{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

If $\lambda_2 = 5$ and $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $A\,\boldsymbol{v} = \lambda_2\,\boldsymbol{v} \iff \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 5 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff$

$$\begin{matrix} 2v_1 + v_2 = 5v_1 \\ 3v_1 + 4v_2 = 5v_2 \end{matrix} \iff v_2 = 3v_1 \Longrightarrow v_1 = 1, \ v_2 = 3 \Longrightarrow \boldsymbol{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- The general solution of the differential equation system is:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = P\,e^{B\,t}\,\boldsymbol{c} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c_1\,e^t \\ c_2\,e^{5t} \end{pmatrix} = \begin{pmatrix} -c_1\,e^t + c_2\,e^{5t} \\ c_1\,e^t + 3\,c_2\,e^{5t} \end{pmatrix}$$

or:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1\,e^{\lambda_1 t}\,\boldsymbol{u} + c_2\,e^{\lambda_2 t}\,\boldsymbol{v} = c_1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}e^{\lambda_1 t} + c_2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}e^{\lambda_2 t} =$$

$$= c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{5t} = \begin{pmatrix} -c_1\,e^t + c_2\,e^{5t} \\ c_1\,e^t + 3\,c_2\,e^{5t} \end{pmatrix}$$

The general solution: $x_1(t) = -c_1\,e^t + c_2\,e^{5t}$
$x_2(t) = c_1\,e^t + 3\,c_2\,e^{5t}$

From the initial conditions: $\begin{matrix} x_1(0) = 3 \\ x_2(0) = 1 \end{matrix} \Longrightarrow \begin{matrix} -c_1 + c_2 = 3 \\ c_1 + 3\,c_2 = 1 \end{matrix} \Longrightarrow c_1 = -2, \ c_2 = 1$

The solution of the initial value problem: $x_1(t) = 2\,e^t + e^{5t}$
$x_2(t) = -2\,e^t + 3\,e^{5t}$

2. $x_1{}' = 3\,x_1 - x_2 \quad x_1(0) = 2$
   $x_2{}' = 4\,x_1 - x_2 \quad x_2(0) = 3$

Solution. The coefficient matrix is $A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$.

• The eigenvalues of A:

$$\det(A - \lambda\, I_2) = \det\begin{pmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix} = (3-\lambda)\,(-1-\lambda) - (-4) = \lambda^2 - 2\,\lambda + 1 = (\lambda - 1)^2 = 0$$

$\Longrightarrow\ \lambda_1 = \lambda_2 = 1$

A has a double real eigenvalue, this is case 2) above.

• The eigenvectors of A:

If $\lambda := \lambda_1 = \lambda_2 = 1$ and $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ then $A\,\boldsymbol{u} = \lambda\,\boldsymbol{u} \iff \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 1\cdot\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \iff$

$$\begin{matrix} 3\,u_1 - u_2 = u_1 \\ 4\,u_1 - u_2 = u_2 \end{matrix} \iff 2\,u_1 = u_2 \implies u_1 = 1,\ u_2 = 2 \implies \boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

If $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $A\,\boldsymbol{v} = \lambda\,\boldsymbol{v} + \boldsymbol{u} \iff \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \iff$

$$\begin{matrix} 3\,v_1 - v_2 = v_1 + 1 \\ 4\,v_1 - v_2 = v_2 + 2 \end{matrix} \iff 2\,v_1 = v_2 + 1 \implies v_1 = 1,\ v_2 = 1 \implies \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

• The general solution of the differential equation system is:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = P\,e^{B\,t}\,\boldsymbol{c} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\cdot e^{\lambda\,t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\cdot e^t\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\cdot e^t\begin{pmatrix} c_1 + t\,c_2 \\ c_2 \end{pmatrix} = e^t\begin{pmatrix} 1\cdot(c_1 + t\,c_2) + 1\cdot c_2 \\ 2\cdot(c_1 + t\,c_2) + 1\cdot c_2 \end{pmatrix}$$

The general solution: $x_1(t) = (c_1 + c_2)\,e^t + c_2\,t\,e^t$
$x_2(t) = (2\,c_1 + c_2)\,e^t + 2\,c_2\,t\,e^t$

From the initial conditions: $\begin{matrix} x_1(0) = 2 \\ x_2(0) = 3 \end{matrix} \implies \begin{matrix} c_1 + c_2 = 2 \\ 2\,c_1 + c_2 = 3 \end{matrix} \implies c_1 = 1,\ c_2 = 1$

The solution of the initial value problem: $x_1(t) = 2\,e^t + t\,e^t$
$x_2(t) = 3\,e^t + 2\,t\,e^t$

3. $x_1{}' = 2\,x_1 + x_2 \quad x_1(0) = 4$
   $x_2{}' = -x_1 + 2\,x_2 \quad x_2(0) = 5$

Solution. The coefficient matrix is $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.

- The eigenvalues of A:

$$\det(A - \lambda I_2) = \det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda) - (-1) = \lambda^2 - 4\lambda + 5 = 0$$

$$\Longrightarrow \lambda_{1,2} = 2 \pm i$$

A has complex eigenvalues, this is case 3) above.

- The eigenvectors of A:

Let $\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be the complex eigenvector corresponding to $\lambda_1 = 2 + i$. Then

$$A\,\boldsymbol{p} = \lambda_1 \boldsymbol{p} \iff \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (2+i)\cdot\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \iff \begin{matrix} 2p_1 + p_2 = (2+i)\,p_1 \\ -p_1 + 2p_2 = (2+i)\,p_2 \end{matrix} \iff \begin{matrix} p_2 = i\,p_1 \\ -p_1 = i\,p_2 \end{matrix} \iff p_2 = i\,p_1$$

$$\Longrightarrow p_1 = 1,\ p_2 = i \Longrightarrow \boldsymbol{p} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 + 0\cdot i \\ 0 + 1\cdot i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Longrightarrow$ The real and imaginary parts of $\boldsymbol{p}$ are: $\boldsymbol{u} = \mathrm{Re}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{v} = \mathrm{Im}(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- The general solution of the differential equation system is $(\lambda_{1,2} = 2 \pm i = \alpha \pm \beta\, i)$:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = P\,e^{B\,t}\,\boldsymbol{c} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{2t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{2t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1\cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix} = e^{2t}\begin{pmatrix} c_1\cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}$$

The general solution: $x_1(t) = c_1\,e^{2t}\cos t + c_2\,e^{2t}\sin t$

$x_2(t) = -c_1\,e^{2t}\sin t + c_2\,e^{2t}\cos t$

From the initial conditions: $\begin{matrix} x_1(0) = 4 \\ x_2(0) = 5 \end{matrix} \Longrightarrow \begin{matrix} c_1 + 0 = 4 \\ 0 + c_2 = 5 \end{matrix} \Longrightarrow c_1 = 4,\ c_2 = 5$

The solution of the initial value problem: $x_1(t) = 4\,e^{2t}\cos t + 5\,e^{2t}\sin t$

$x_2(t) = -4\,e^{2t}\sin t + 5\,e^{2t}\cos t$

Remark. If the eigenvalues are complex, $\lambda_{1,2} = \alpha \pm \beta\, i$, and $\boldsymbol{p}$ is an eigenvector corresponding to λ_1 then $\boldsymbol{x}_{\text{complex}}(t) = e^{\lambda_1 t}\,\boldsymbol{p}$ is a complex solution.
It can be shown that $\mathrm{Re}(\boldsymbol{x}_{\text{complex}}(t))$ and $\mathrm{Im}(\boldsymbol{x}_{\text{complex}}(t))$ are also solutions and linearly independent. Since the dimension of the linear space generated by the solutions has dimension 2 then these two functions constitute a basis, so the general solution of the differential equation system is $\boldsymbol{x}(t) = c_1\,\mathrm{Re}(\boldsymbol{x}_{\text{complex}}(t)) + c_2\,\mathrm{Im}(\boldsymbol{x}_{\text{complex}}(t))$.

In the above example $\lambda_{1,2} = 2 \pm i$, the eigenvector corresponding to $\lambda_1 = 2 + i$ is $\boldsymbol{p} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Using that $e^{i\,t} = \cos t + i \sin t$, a complex solution is

$$\boldsymbol{x}_{\text{complex}}(t) = e^{\lambda_1 t}\,\boldsymbol{p} = e^{(2+i)\,t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{2\,t}(\cos t + i \sin t)\begin{pmatrix} 1 \\ i \end{pmatrix} =$$

$$= e^{2\,t}\begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix} = e^{2\,t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \cdot e^{2\,t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

The real and imaginary parts of the complex solution are

$$\operatorname{Re}(\boldsymbol{x}_{\text{complex}}(t)) = e^{2\,t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \text{ and } \operatorname{Im}(\boldsymbol{x}_{\text{complex}}(t)) = e^{2\,t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

so the general solution of the differential equation system is:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1\, e^{2\,t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\, e^{2\,t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\Longrightarrow x_1(t) = c_1\, e^{2\,t} \cos t + c_2\, e^{2\,t} \sin t$$

$$x_2(t) = -c_1\, e^{2\,t} \sin t + c_2\, e^{2\,t} \cos t$$

Homework

Solve the following initial value problems:

1. $x_1' = -x_1 + 8\,x_2$ $\quad x_1(0) = 7$
   $x_2' = x_1 + x_2$ $\quad x_2(0) = -1$

2. $x_1' = 5\,x_1 - x_2$ $\quad x_1(0) = 4$
   $x_2' = 2\,x_1 + 3\,x_2$ $\quad x_2(0) = -1$

3. $x_1' = 2\,x_1 + 6\,x_2$ $\quad x_1(0) = 5$
   $x_2' = x_1 + x_2$ $\quad x_2(0) = 0$

4. $x_1' = 3\,x_1 + 4\,x_2$ $\quad x_1(0) = 1$
   $x_2' = -4\,x_1 + 3\,x_2$ $\quad x_2(0) = 2$

5. $x_1' = x_2$ $\quad x_1(0) = 3$
   $x_2' = -x_1$ $\quad x_2(0) = 7$

6. The population of a species in two habitats is described by the functions $x(t)$ and $y(t)$.
   If the increase of the population is described by the simple growth model in the first habitat and there is migration in both directions that is proportional to the population, then we obtain the system

   $$x' = a\,x + b\,y$$
   $$y' = r\,x$$

   Find the general solution if $a = r = 1$ and $b = 2$. Show that after a long time, the ratio of the populations in the two habitats will be constant.

7. In the previous model, if the increase of the population is described by the simple growth model in the second habitat as well, then we obtain the system

   $$x' = a\,x + b\,y$$
   $$y' = r\,x + s\,y$$

   Find the general solution if $a = r = s = 1$ and $b = 2$.

Solutions

1. $x_1{}' = -x_1 + 8\,x_2 \quad x_1(0) = 7$
$x_2{}' = x_1 + x_2 \quad x_2(0) = -1$

Solution. The coefficient matrix is $A = \begin{pmatrix} -1 & 8 \\ 1 & 1 \end{pmatrix}$.

• The eigenvalues of A:

$$\det(A - \lambda\, I_2) = \det\begin{pmatrix} -1-\lambda & 8 \\ 1 & 1-\lambda \end{pmatrix} = (-1-\lambda)\,(1-\lambda) - 8 = \lambda^2 - 9 = (\lambda-3)\,(\lambda+3) = 0$$

$$\Longrightarrow \lambda_1 = 3,\ \ \lambda_2 = -3$$

• The eigenvectors of A:

If $\lambda_1 = 3$ and $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ then $A\,\boldsymbol{u} = \lambda_1\,\boldsymbol{u} \iff \begin{pmatrix} -1 & 8 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 3\cdot\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \iff$

$$\begin{matrix} -u_1 + 8\,u_2 = 3\,u_1 \\ u_1 + u_2 = 3\,u_2 \end{matrix} \iff u_1 = 2\,u_2 \implies u_2 = 1,\ u_1 = 2 \implies \boldsymbol{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

If $\lambda_2 = -3$ and $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $A\,\boldsymbol{v} = \lambda_2\,\boldsymbol{v} \iff \begin{pmatrix} -1 & 8 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -3\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff$

$$\begin{matrix} -v_1 + 8\,v_2 = -3\,v_1 \\ v_1 + v_2 = -3\,v_2 \end{matrix} \iff v_1 = -4\,v_2 \implies v_2 = 1,\ v_1 = -4 \implies \boldsymbol{v} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

• The general solution of the differential equation system is:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = P\,e^{B\,t}\,\boldsymbol{c} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1\,e^{3t} \\ c_2\,e^{-3t} \end{pmatrix} = \begin{pmatrix} 2\,c_1\,e^{3t} - 4\,c_2\,e^{-3t} \\ c_1\,e^{3t} + c_2\,e^{-3t} \end{pmatrix}$$

or:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1\,e^{\lambda_1 t}\,\boldsymbol{u} + c_2\,e^{\lambda_2 t}\,\boldsymbol{v} = c_1\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}e^{\lambda_1 t} + c_2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}e^{\lambda_2 t} =$$

$$c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} -4 \\ 1 \end{pmatrix}e^{-3t} = \begin{pmatrix} 2\,c_1\,e^{3t} - 4\,c_2\,e^{-3t} \\ c_1\,e^{3t} + c_2\,e^{-3t} \end{pmatrix}$$

The general solution: $x_1(t) = 2\,c_1\,e^{3t} - 4\,c_2\,e^{-3t}$
$x_2(t) = c_1\,e^{3t} + c_2\,e^{-3t}$

From the initial conditions: $\begin{matrix} x_1(0) = 7 \\ x_2(0) = -1 \end{matrix} \implies \begin{matrix} 2\,c_1 - 4\,c_2 = 7 \\ c_1 + c_2 = -1 \end{matrix} \implies c_1 = \frac{1}{2},\ c_2 = -\frac{3}{2}$

The solution of the initial value problem: $x_1(t) = e^{3t} + 6\,e^{-3t}$

$$x_2(t) = \frac{1}{2}\,e^{3t} - \frac{3}{2}\,e^{-3t}$$

2. $x_1' = 5\,x_1 - x_2 \qquad x_1(0) = 4$

$\quad x_2' = 2\,x_1 + 3\,x_2 \qquad x_2(0) = -1$

Solution. The coefficient matrix is $A = \begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$.

- The eigenvalues of A:

$$\det(A - \lambda I_2) = \det\begin{pmatrix} 5-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix} = (5-\lambda)(3-\lambda) + 2 = \lambda^2 - 8\lambda + 17 = 0$$

$$\Longrightarrow \lambda_{1,2} = \frac{8 \pm \sqrt{64 - 4\cdot 17}}{2} = \frac{8 \pm 2\,i}{2} = 4 \pm i$$

- The eigenvectors of A:

Let $\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ be the complex eigenvector corresponding to $\lambda_1 = 4 + i$. Then

$$A\,\boldsymbol{p} = \lambda_1\,\boldsymbol{p} \iff \begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = (4+i)\cdot\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \iff \begin{matrix} 5\,p_1 - p_2 = (4+i)\,p_1 \\ 2\,p_1 + 3\,p_2 = (4+i)\,p_2 \end{matrix} \iff \begin{matrix} p_1 - p_2 = i\,p_1 \\ 2\,p_1 = p_2 + i\,p_2 \end{matrix}$$

$$\iff \begin{matrix} p_1(1-i) = p_2 \\ 2\,p_1 = (1+i)\,p_2 \end{matrix} \Longrightarrow p_1 = 1+i,\ p_2 = 2 \Longrightarrow \boldsymbol{p} = \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \begin{pmatrix} 1+1\cdot i \\ 2+0\cdot i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or: $p_1 = 1,\ p_2 = 1 - i$...

$\Longrightarrow$ The real and imaginary parts of $\boldsymbol{p}$ are: $\boldsymbol{u} = \mathrm{Re}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\boldsymbol{v} = \mathrm{Im}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- The general solution of the differential equation system is $(\lambda_{1,2} = 4 \pm i = \alpha \pm \beta\,i)$:

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = P\,e^{Bt}\,\boldsymbol{c} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} e^{4t}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} =$$

$$= e^{4t}\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} c_1\cos t + c_2\,\mathrm{sint} \\ -c_1\sin t + c_2\cos t \end{pmatrix} = e^{4t}\begin{pmatrix} (c_1\cos t + c_2\,\mathrm{sint}) + (-c_1\sin t + c_2\cos t) \\ 2\,(c_1\cos t + c_2\,\mathrm{sint}) \end{pmatrix}$$

The general solution: $x_1(t) = (c_1 + c_2)\,e^{4t}\cos t + (-c_1 + c_2)\,e^{4t}\sin t$

$$x_2(t) = 2\,c_1\,e^{4t}\cos t + 2\,c_2\,e^{4t}\sin t$$

Or: A complex solution is

$$\boldsymbol{x}_{\text{complex}}(t) = e^{\lambda_1 t}\,\boldsymbol{p} = e^{(4+i)t}\begin{pmatrix} 1+i \\ 2 \end{pmatrix} = e^{4t}(\cos t + i\sin t)\begin{pmatrix} 1+i \\ 2 \end{pmatrix} = e^{4t}\begin{pmatrix} (\cos t + i\sin t)(1+i) \\ (\cos t + i\sin t)\cdot 2 \end{pmatrix} =$$

$$= e^{4t}\begin{pmatrix} (\cos t - \sin t) + i(\cos t + \sin t) \\ 2\cos t + i \cdot 2\sin t \end{pmatrix} = e^{4t}\begin{pmatrix} \cos t - \sin t \\ 2\cos t \end{pmatrix} + i \cdot e^{4t}\begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix}$$

The real and imaginary parts of the complex solution are:

$$\operatorname{Re}(\boldsymbol{x}_{\text{complex}}(t)) = e^{4t}\begin{pmatrix} \cos t - \sin t \\ 2\cos t \end{pmatrix}$$

$$\operatorname{Im}(\boldsymbol{x}_{\text{complex}}(t)) = e^{4t}\begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix}$$

The general solution is

$$\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{4t}\begin{pmatrix} \cos t - \sin t \\ 2\cos t \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix} \Longrightarrow$$

$$x_1(t) = c_1 e^{4t}(\cos t - \sin t) + c_2 e^{4t}(\cos t + \sin t) = (c_1 + c_2) e^{4t} \cos t + (-c_1 + c_2) e^{4t} \sin t$$
$$x_2(t) = 2 c_1 e^{4t} \cos t + 2 c_2 e^{4t} \sin t$$

From the initial conditions: $\begin{matrix} x_1(0) = 4 \\ x_2(0) = -1 \end{matrix} \Longrightarrow \begin{matrix} (c_1 + c_2) + 0 = 4 \\ 2 c_1 + 0 = -1 \end{matrix} \Longrightarrow c_1 = -\frac{1}{2},\ c_2 = \frac{9}{2}$

The solution of the initial value problem: $\begin{matrix} x_1(t) = 4 e^{4t} \cos t + 5 e^{4t} \sin t \\ x_2(t) = -e^{4t} \cos t + 9 e^{4t} \sin t \end{matrix}$

Results

3. $x_1{}' = 2 x_1 + 6 x_2 \quad x_1(0) = 5$
   $x_2{}' = x_1 + x_2 \quad x_2(0) = 0$

Result: $x_1(t) = 2 e^{-t} + 3 e^{4t}$
$x_2(t) = -e^{-t} + e^{4t}$

4. $x_1{}' = 3 x_1 + 4 x_2 \quad x_1(0) = 1$
   $x_2{}' = -4 x_1 + 3 x_2 \quad x_2(0) = 2$

Result: $x_1(t) = e^{3t} \cos 4t + 2 e^{3t} \sin 4t$
$x_2(t) = 2 e^{3t} \cos 4t - e^{3t} \sin 4t$

5. $x_1{}' = x_2 \quad x_1(0) = 3$
   $x_2{}' = -x_1 \quad x_2(0) = 7$

Result: $x_1(t) = 3 \cos t + 7 \sin t$
$x_2(t) = 7 \cos t - 3 \sin t$

6. $x' = a x + b y$ if $a = r = 1, b = 2$
$y' = r x$

The general solution is
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 c_1 e^{2t} - c_2 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} \end{pmatrix}$$

The ratio of the populations when $t \longrightarrow \infty$:

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = \lim_{t\to\infty} \frac{2 c_1 e^{2t} - c_2 e^{-t}}{c_1 e^{2t} + c_2 e^{-t}} = \lim_{t\to\infty} \frac{e^{2t}}{e^{2t}} \cdot \frac{2 c_1 - c_2 e^{-3t}}{c_1 + c_2 e^{-3t}} = \lim_{t\to\infty} \frac{2 c_1 - c_2 e^{-3t}}{c_1 + c_2 e^{-3t}} = \frac{2 c_1}{c_1} = 2$$

using that $\lim_{t\to\infty} e^{-3t} = 0$. It means that after a long time the population in the first habitat will be approximately twice the population in the second habitat.

7. $x' = a x + b y$ if $a = r = s = 1, b = 2$
$y' = r x + s y$

The general solution
is
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(1+\sqrt{2})t} & 0 \\ 0 & e^{(1-\sqrt{2})t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\, c_1 e^{(1+\sqrt{2})t} - \sqrt{2}\, c_2 e^{(1-\sqrt{2})t} \\ c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t} \end{pmatrix}$$