07 - Power series method

Recall that if f is at least n times differentiable then the Taylor polynomial of order n generated by $f(x)$ about x_0 is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

With the power series method the solution $y(x)$ of an initial value problem about x_0 can be expressed in the form of a Taylor series about x_0 if from the differential equation a recursion can be obtained for the coefficients. The solution is usually only approximated by a Taylor polynomial for which we calculate the first few coefficients.

In the exercises $x_0 = 0$, then the solution can be approximated as

$$y(x) \approx T_n(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots + \frac{y^{(n)}(0)}{n!}x^n.$$

We obtain the values $y(0),\ y'(0),\ y''(0),$ etc. from the given data and calculate them by differentiating the equation repeatedly. Then these values are substituted into the Taylor polynomial.

Exercises

1. $y'(x) - 2\,x\,y(x) = 1,\ \ y(0) = 1.$
Find the 4th degree Taylor polynomial of the solution to the differential equation.

2. $y''(x) = y^2(x),\ \ y(0) = y'(0) = 1.$
Find the 3rd degree Taylor polynomial of the solution to the differential equation.

3. $y'(t) - y(t)\sin t = t,\ \ y(0) = 2.$
Find the 3rd degree Taylor polynomial of the solution to the differential equation.

Solutions

1. $y'(x) - 2\,x\,y(x) = 1,\ \ y(0) = 1.$
Find the 4th degree Taylor polynomial of the solution to the differential equation.

Solution. For the Taylor polynomial we need the values $y(0),\ y'(0),\ y''(0),\ y'''(0),\ y^{(4)}(0).$
The equation is $y'(x) = 2\,x\,y(x) + 1.$

1) From the initial condition $y(0) = 1$

2) Substituting $x = 0$ into the equation: $y'(0) = 2 \cdot 0 \cdot y(0) + 1 = 1 \implies y'(0) = 1$

The other values of the derivatives can be calculated by differentiating the equation repeatedly, recursively using previously values calculated at $x = 0$.

3) Differentiating both sides of $y'(x) = 2\,x\,y(x) + 1$

$\implies y''(x) = 2\,y(x) + 2\,x\,y'(x) + 0$

$\implies y''(0) = 2\,y(0) + 2 \cdot 0 \cdot y'(0) = 2 \cdot 1 + 2 \cdot 0 \cdot 1 = 2 \implies y''(0) = 2$

4) Differentiating both sides of $y''(x) = 2\,y(x) + 2\,x\,y'(x)$

$\implies y'''(x) = 2\,y'(x) + 2\,y'(x) + 2\,x\,y''(x) = 4\,y'(x) + 2\,x\,y''(x)$

$\implies y'''(0) = 4\,y'(0) + 2 \cdot 0 \cdot y''(0) = 4 \cdot 1 + 2 \cdot 0 \cdot 2 = 4 \implies y'''(0) = 4$

5) Differentiating both sides of $y'''(x) = 4\,y'(x) + 2\,x\,y''(x)$

$\implies y^{(4)}(x) = 4\,y''(x) + 2\,y''(x) + 2\,x\,y'''(x) = 6\,y''(x) + 2\,x\,y'''(x)$

$\implies y^{(4)}(0) = 6\,y''(0) + 2 \cdot 0 \cdot y'''(0) = 6 \cdot 2 + 2 \cdot 0 \cdot 4 = 12 \implies y^{(4)}(0) = 12$

We have $y(0) = 1,\ y'(0) = 1,\ y''(0) = 2,\ y'''(0) = 4,\ y^{(4)}(0) = 12$, so the approximation of the solution is

$$y(x) \approx T_4(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}\,x^2 + \frac{y'''(0)}{3!}\,x^3 + \frac{y^{(4)}(0)}{4!}\,x^4 =$$

$$= 1 + 1 \cdot x + \frac{2}{2!}\,x^2 + \frac{4}{3!}\,x^3 + \frac{12}{4!}\,x^4 = 1 + x + x^2 + \frac{2}{3}\,x^2 + \frac{1}{2}\,x^4$$

2. $y''(x) = y^2(x),\ \ y(0) = y'(0) = 1.$

Find the 3rd degree Taylor polynomial of the solution to the differential equation.

Solution. For the Taylor polynomial we need the values $y(0),\ y'(0),\ y''(0),\ y'''(0)$.
The equation is $y'(x) = 2\,x\,y(x) + 1$.

1) From the initial condition $y(0) = y'(0) = 1$

2) Substituting $x = 0$ into the equation: $y''(0) = y^2(0) = 1^2 = 1 \implies y''(0) = 1$

3) Differentiating both sides of $y''(x) = y^2(x)$ using the chain rule:

$y''(x) = (y(x))^2 \implies y'''(x) = 2\,y(x)\,y'(x)$

$\implies y'''(0) = 2\,y(0)\,y'(0) = 2 \cdot 1 \cdot 1 = 2 \implies y'''(0) = 2$

The approximation of the solution is

$$y(x) \approx T_3(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}\,x^2 + \frac{y'''(0)}{3!}\,x^3 =$$

$$= 1 + 1\cdot x + \frac{1}{2!}\,x^2 + \frac{2}{3!}\,x^3 = 1 + x + \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3$$

3. $y'(t) - y(t)\sin t = t,\ \ y(0) = 2.$
Find the 3rd degree Taylor polynomial of the solution to the differential equation.

Solution. For the Taylor polynomial we need the values $y(0),\ y'(0),\ y''(0),\ y'''(0)$.
The equation is $y'(t) = t + y(t)\sin t$.

1) From the initial condition $y(0) = 2$
2) Substituting $t = 0$ into the equation: $y'(0) = 0 + y(0)\sin 0 = 0 + 2\cdot 0 = 0 \implies y'(0) = 0$

3) Differentiating both sides of $y'(t) = t + y(t)\sin t$
$\implies\ y''(t) = 1 + y'(t)\sin t + y(t)\cos t$
$\implies\ y''(0) = 1 + y'(0)\sin 0 + y(0)\cos 0 = 1 + 0\cdot 0 + 2\cdot 1 = 3 \implies y''(0) = 3$

4) Differentiating both sides of $y''(t) = 1 + y'(t)\sin t + y(t)\cos t$
$\implies\ y'''(t) = y''(t)\sin t + y'(t)\cos t + y'(t)\cos t + y(t)(-\sin t)$
$\implies\ y'''(0) = 3\cdot 0 + 0\cdot 1 + 0\cdot 1 + 2\cdot 0 = 0 \implies y'''(0) = 0$

The approximation of the solution is

$$y(t) \approx T_3(t) = y(0) + y'(0)\,t + \frac{y''(0)}{2!}\,t^2 + \frac{y'''(0)}{3!}\,t^3 =$$

$$= 2 + 0\cdot t + \frac{3}{2!}\,t^2 + \frac{0}{3!}\,t^3 = 2 + \frac{3}{2}\,t^2$$

Homework

1. $y''(x) = x\,y^2(x) - y'(x),\ \ y(0) = 2,\ \ y'(0) = 1.$
Find the 3rd degree Taylor polynomial of the solution to the differential equation.

2. The angle of a pendulum of length l is given by $\theta(t)$ as a function of the time.
From the equation of motion, dividing by m, we obtain $\theta''(t) + \frac{g}{l}\sin\theta(t) = 0$.
Let $\omega := \sqrt{\frac{g}{l}}$. If the pendulum is pushed from vertical position with velocity v, then the initial conditions are $\theta(0) = 0,\ \ \theta'(0) = v$. Find the 3rd degree Taylor polynomial of the solution to the differential equation.

3. In the chemical reaction $X + Y \longrightarrow 2X$, the concentrations of the species X and Y are described by $x(t)$ and $y(t)$ as a function of the time. The reaction can be modelled by the differential equation system

$x'(t) = k\,x(t)\,y(t)$

$y'(t) = -k\,x(t)\,y(t)$

where $k > 0$ is the reaction rate coefficient. Let $k = 1$ and $x(0) = y(0) = 1$.
Find the 3rd degree Taylor polynomial of the functions $x(t)$ and $y(t)$.

Solutions

1. $y''(x) = x\,y^2(x) - y'(x),\; y(0) = 2,\; y'(0) = 1.$
Find the 3rd degree Taylor polynomial of the solution to the differential equation.

Solution. For the Taylor polynomial we need the values $y(0),\; y'(0),\; y''(0),\; y'''(0)$.

1) From the initial conditions $y(0) = 2,\; y'(0) = 1$

2) Substituting $x = 0$ into the equation:

$$y''(0) = 0 \cdot y^2(0) - y'(0) = 0 \cdot 2^2 - 1 = -1$$

3) Differentiating both sides of $y''(x) = x\,y^2(x) - y'(x)$

$\Longrightarrow\; y'''(x) = y^2(x) + x \cdot 2\,y(x)\,y'(x) - y''(x)$

$\Longrightarrow\; y'''(0) = 2^2 + 0 \cdot 2 \cdot 2 \cdot 1 - (-1) = 5$

The approximation of the solution is

$$y(x) \approx T_3(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}\,x^2 + \frac{y'''(0)}{3!}\,x^3 =$$

$$= 2 + 1 \cdot x + \frac{-1}{2!}\,x^2 + \frac{5}{3!}\,x^3 = 2 + x - \frac{1}{2}\,x^2 + \frac{5}{6}\,x^3$$

2. The angle of a pendulum of length L is given by $\theta(t)$ as a function of the time.
From the equation of motion, dividing by m, we obtain $\theta''(t) + \frac{g}{L}\sin\theta(t) = 0$.

Let $\omega := \sqrt{\frac{g}{L}}$. If the pendulum is pushed from vertical position with velocity v, then the initial conditions are $\theta(0) = 0,\; \theta'(0) = v$. Find the 3rd degree Taylor polynomial of the solution to the differential equation.

Solution. For the Taylor polynomial we need the values $\theta(0),\; \theta'(0),\; \theta''(0),\; \theta'''(0)$.

The equation is $\theta''(t) = -\frac{g}{L}\sin\theta(t)$.

1) From the initial conditions $\theta(0) = 0,\; \theta'(0) = v$

2) Substituting $t = 0$ into the equation:

$$\theta''(0) = -\frac{g}{L}\sin\theta(0) = 0$$

3) Differentiating both sides of $\theta''(t) = -\frac{g}{L}\sin(\theta(t))$ using the chain rule:

$$\Longrightarrow\; \theta'''(t) = -\frac{g}{L}\cos(\theta(t)) \cdot \theta'(t)$$

$$\Longrightarrow \quad \theta'''(0) = -\frac{g}{L}\cos(\theta(0))\cdot\theta'(0) = -\frac{g}{L}\cos(0)\cdot\theta'(0) = -\frac{g}{L}\cdot 1\cdot v = -\omega^2 v$$

The approximation of the solution is

$$\theta(t) \approx T_3(t) = \theta(0) + \theta'(0)\,t + \frac{\theta''(0)}{2!}\,t^2 + \frac{\theta'''(0)}{3!}\,t^3 =$$

$$= 0 + v\,t + \frac{0}{2!}\,t^2 - \frac{\omega^2 v}{3!}\,t^3 = v\,t - \frac{\omega^2 v}{6}\,t$$

3. In the chemical reaction $X + Y \longrightarrow 2X$, the concentrations of the species X and Y are described by $x(t)$ and $y(t)$ as a function of the time. The reaction can be modelled by the differential equation system

$$x'(t) = k\,x(t)\,y(t)$$
$$y'(t) = -k\,x(t)\,y(t)$$

where $k > 0$ is the reaction rate coefficient. Let $k = 1$ and $x(0) = y(0) = 1$.
Find the 3rd degree Taylor polynomial of the functions $x(t)$ and $y(t)$.

Solution.

1) From the initial conditions $x(0) = y(0) = 1$

2) From the equation system the first derivatives and the substitution values:

$$x'(t) = x(t)\,y(t) \quad \Longrightarrow \quad x'(0) = x(0)\,y(0) = 1\cdot 1 = 1$$
$$y'(t) = -x(t)\,y(t) \qquad\qquad y'(0) = -x(0)\,y(0) = -1\cdot 1 = -1$$

3) The second derivatives and the substitution values:

$$x''(t) = x'(t)\,y(t) + x(t)\,y'(t) \quad \Longrightarrow \quad x''(0) = x'(0)\,y(0) + x(0)\,y'(0) = 1\cdot 1 + 1\cdot(-1) = 0$$
$$y''(t) = -x'(t)\,y(t) - x(t)\,y'(t) \qquad\qquad y''(0) = -x'(0)\,y(0) - x(0)\,y'(0) = -1\cdot 1 - 1\cdot(-1) = 0$$

4) The third derivatives and the substitution values:

$$x'''(t) = x''(t)\,y(t) + x'(t)\,y'(t) + x'(t)\,y'(t) + x(t)\,y''(t)$$
$$y'''(t) = -x''(t)\,y(t) - x'(t)\,y'(t) - x'(t)\,y'(t) - x(t)\,y''(t)$$

$$\Longrightarrow \quad x''(0) = 0\cdot 1 + 1\cdot(-1) + 1\cdot(-1) + 1\cdot 0 = -2$$
$$y''(0) = -0\cdot 1 - 1\cdot(-1) - 1\cdot(-1) - 1\cdot 0 = 2$$

The approximations of the solutions are

$$x(t) \approx x(0) + x'(0)\,t + \frac{x''(0)}{2!}\,t^2 + \frac{x'''(0)}{3!}\,t^3 = 1 + 1\cdot t + \frac{0}{2!}\,t^2 + \frac{-2}{3!}\,t^3 = 1 + t - \frac{1}{3}\,t^3$$

$$y(t) \approx y(0) + y'(0)\,t + \frac{y''(0)}{2!}\,t^2 + \frac{y'''(0)}{3!}\,t^3 = 1 - 1\cdot t + \frac{0}{2!}\,t^2 + \frac{2}{3!}\,t^3 = 1 - t + \frac{1}{3}\,t^3$$