06 - Laplace transforms

1.

Definition. Let $f(t)$ be a given function that is defined for all $t \geq 0$. The Laplace transform of $f(t)$ is

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt \text{ if the integral exists.}$$

Exercise 1. Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$.

2. Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant. Find $F(s)$.

Solution:

1. $F(s) = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{A\to\infty} \int_0^A e^{-st}\, dt = \lim_{A\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^A = \lim_{A\to\infty} \left(\frac{e^{-sA}}{-s} - \frac{1}{-s}\right) = 0 - \frac{1}{-s} = \frac{1}{s}$

2. $F(s) = \int_0^\infty e^{-st} e^{at}\, dt = \lim_{A\to\infty} \int_0^A e^{(a-s)t}\, dt = \lim_{A\to\infty} \left[\frac{e^{(a-s)t}}{a-s}\right]_0^A = \lim_{A\to\infty} \left(\frac{e^{(a-s)A}}{a-s} - \frac{1}{a-s}\right) = 0 - \frac{1}{a-s} = \frac{1}{s-a}$

Theorem (Existence theorem for Laplace transforms).
Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t > 0$ and satisfies $|f(t)| \leq M e^{kt}$ for all $t > 0$ and for some constants k and M.
Then the Laplace transform of $f(t)$ exists for all $s > k$.

2. Properties of the Laplace transform

Let $F(s)$ and $G(s)$ denote the Laplace transform of $f(t)$ and $g(t)$, respectively. Then

1. $\mathcal{L}\{a f(t) + b g(t)\}(s) = a F(s) + b G(s) \qquad (a, b \in \mathbb{R})$
2. $\mathcal{L}\{e^{at} f(t)\}(s) = F(s-a)$
3. $\mathcal{L}\{f(at)\}(s) = \frac{1}{a} F\left(\frac{s}{a}\right)$
4. $\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(r)\, dr$
5. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s)$
6. $\mathcal{L}\{f'(t)\}(s) = s F(s) - f(0)$
7. $\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - s f(0) - f'(0)$
8. $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$

Proof of 6.: With integration by parts:

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty f'(t) e^{-st}\, dt = \left[f(t) e^{-st}\right]_0^\infty - \int_0^\infty f(t)(-s) e^{-st}\, dt =$$

$$= (0 - f(0)) + s \int_0^\infty f(t) e^{-st}\, dt = s F(s) - f(0)$$

3.

Some functions and their Laplace transforms ($n \in \mathbb{N}$, $a, b \in \mathbb{R}$).

□	$f(t)$	$\mathcal{L}\{f(t)\}(s)$	□	$f(t)$	$\mathcal{L}\{f(t)\}(s)$
1	1	$\frac{1}{s}$	2	t^n	$\frac{n!}{s^{n+1}}$
3	$e^{a t}$	$\frac{1}{s-a}$	4	$t^n e^{a t}$	$\frac{n!}{(s-a)^{n+1}}$
5	$\sin(a t)$	$\frac{a}{s^2+a^2}$	6	$\cos(a t)$	$\frac{s}{s^2+a^2}$
7	$t \sin(a t)$	$\frac{2 a s}{(s^2+a^2)^2}$	8	$t \cos(a t)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
9	$e^{b t} \sin(a t)$	$\frac{a}{(s-b)^2+a^2}$	10	$e^{b t} \cos(a t)$	$\frac{s-b}{(s-b)^2+a^2}$
11	$\operatorname{sh}(a t)$	$\frac{a}{s^2-a^2}$	12	$\operatorname{ch}(a t)$	$\frac{s}{s^2-a^2}$