06 - Laplace transforms, solutions

1. Laplace transform

a) $f(t) = 7\sin 3t \quad \Longrightarrow \quad F(s) = 7\dfrac{3}{s^2+3^2} = \dfrac{21}{s^2+9}$

b) $f(t) = 6t^2 + 3t - 2 \quad \Longrightarrow \quad F(s) = 6\cdot\dfrac{2!}{s^{2+1}} + 3\cdot\dfrac{1!}{s^{1+1}} - 2\cdot\dfrac{1}{s} = \dfrac{12}{s^3} + \dfrac{3}{s^2} - \dfrac{2}{s}$

c) $f(t) = t\cos 7t \quad \Longrightarrow \quad F(s) = \dfrac{s^2-7^2}{\left(s^2+7^2\right)^2} = \dfrac{s^2-49}{\left(s^2+49\right)^2}$

d) $f(t) = e^{2t}\sin 3t \quad \Longrightarrow \quad F(s) = \dfrac{3}{(s-2)^2+3^2} = \dfrac{3}{(s-2)^2+9}$

e)* $f(t) = t\,e^{-t}\cos 4t \quad \Longrightarrow \quad F(s) = (-1)\cdot\left(\dfrac{s+1}{(s+1)^2+16}\right)' = \dfrac{-15+2s+s^2}{\left(17+2s+s^2\right)^2}$

f)* $f(t) = t^2\sin 5t \quad \Longrightarrow \quad F(s) = (-1)^2\cdot\left(\dfrac{5}{s^2+25}\right)'' = \dfrac{10\left(-25+3s^2\right)}{\left(25+s^2\right)^3}$

For e) and f) we need property 5.: $\mathcal{L}\left\{t^n f(t)\right\}(s) = (-1)^n F^{(n)}(s)$

2. Inverse Laplace transform

a) $F(s) = \dfrac{3}{s} + \dfrac{1}{s-5} - \dfrac{7}{s-2}$

$\Longrightarrow \quad f(t) = 3 + e^{5t} - 7e^{2t}$

b) $F(s) = \dfrac{11}{s-3} + \dfrac{4}{s^2-25}$

$F(s) = \dfrac{11}{s-3} + \dfrac{4}{s^2-25} = 11\cdot\dfrac{1}{s-3} + \dfrac{4}{5}\dfrac{5}{s^2-25} \quad \Longrightarrow \quad f(t) = 11e^{3t} + \dfrac{4}{5}\,\mathrm{sh}(5t)$

c) $F(s) = \dfrac{7}{s^2+4}$

$F(s) = \dfrac{7}{s^2+4} = \dfrac{7}{2}\cdot\dfrac{2}{s^2+4} \quad \Longrightarrow \quad f(t) = \dfrac{7}{2}\sin(2t)$

d) $F(s) = \dfrac{s+4}{s^2+9}$

$F(s) = \dfrac{s+4}{s^2+9} = \dfrac{s}{s^2+9} + \dfrac{4}{3}\cdot\dfrac{3}{s^2+9} \quad \Longrightarrow \quad f(t) = \cos(3t) + \dfrac{4}{3}\sin(3t)$

e) $F(s) = \dfrac{3}{s^2+4s+14}$

The denominator has complex roots $\Longrightarrow$ completing the square

$$F(s)=\frac{3}{s^2+4s+14}=\frac{3}{(s+2)^2+10}=\frac{3}{\sqrt{10}}\cdot\frac{\sqrt{10}}{(s-(-2))^2+\left(\sqrt{10}\right)^2}\implies f(t)=\frac{3}{\sqrt{10}}e^{-2t}\sin\left(\sqrt{10}\,t\right)$$

f) $F(s)=\frac{4}{s^2+2s}$

The denominator has real roots $\implies$ partial fraction decomposition

The roots of the denominator are $s_1=0$ and $s_2=-2$ (single real roots)

$$\implies F(s)=\frac{4}{s^2+2s}=\frac{4}{s(s+2)}=\frac{A}{s}+\frac{B}{s+2}=\frac{A(s+2)+Bs}{s(s+2)}$$

$\implies \quad 4=A(s+2)+Bs$

$s=-2$: $\quad 4=0+B(-2)\implies B=-2$

$s=0$: $\quad 4=2A+0\implies A=2$

(It is convenient to substitute the roots of the denominator for s.)

$$\implies F(s)=\frac{2}{s}-\frac{2}{s+2}\implies f(t)=2-2e^{-2t}$$

g) $F(s)=\frac{3}{s^3+2s^2}$

The denominator has real roots $\implies$ partial fraction decomposition

The roots of the denominator are $s_1=s_2=0$ (double real root) and $s_3=-2$ (single real root)

$$\implies F(s)=\frac{3}{s^3+2s^2}=\frac{3}{s^2(s+2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2}=\frac{As(s+2)+B(s+2)+Cs^2}{s^2(s+2)}$$

$\implies \quad 3=As(s+2)+B(s+2)+Cs^2$

$s=0$: $\quad 3=0+2B+0\implies B=\frac{3}{2}$

$s=-2$: $\quad 3=0+0+4C\implies C=\frac{3}{4}$

$s=1$: $\quad 3=3A+3B+C\implies A=1-B-\frac{C}{3}=1-\frac{3}{2}-\frac{1}{4}=-\frac{3}{4}$

(Here the roots of the denominator and an arbitrary number is substituted for s.)

$$\implies F(s)=-\frac{3}{4}\cdot\frac{1}{s}+\frac{3}{2}\cdot\frac{1}{s^2}+\frac{3}{4}\cdot\frac{1}{s+2}\implies f(t)=-\frac{3}{4}+\frac{3}{2}t+\frac{3}{4}e^{-2t}$$

3. Solving initial value problems

a) $y' = y,\ y(0) = 3$

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \implies \mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0)$

The Laplace transform of the differential equation is

$s\,Y(s) - 3 = Y(s)$ (or : $s\,Y - 3 = Y$)

It is an algebraic equation for $Y(s)$.

$$sY(s) - Y(s) = 3 \implies Y(s) = \frac{3}{s-1}$$

The solution of the initial value problem is the inverse Laplace transform of $Y(s)$:

$$Y(s) = \frac{3}{s-1} \implies y(t) = 3\,e^{t}$$

b) $y' = 7\,y,\ y(0) = -1$

Similarly as in case a):

$$s\,Y(s) + 1 = 7\,Y(s) \implies Y(s) = \frac{-1}{s-7} \implies y(t) = -e^{7\,t}$$

c) $y'' = -y,\ y(0) = 0, y'(0) = -2$

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \implies \mathcal{L}\{y''(t)\}(s) = s^2\,Y(s) - s\,y(0) - y'(0)$

The Laplace transform of the differential equation is

$s^2\,Y(s) - s \cdot 0 - (-2) = -Y(s)$ (or : $s^2\,Y - s \cdot 0 - (-2) = -Y$)

$$s^2\,Y(s) + 2 = -Y(s)$$

$$\left(s^2 + 1\right) Y(s) = -2$$

$$Y(s) = -\frac{2}{s^2+1} = -2 \cdot \frac{1}{s^2+1}$$

The solution of the initial value problem is the inverse Laplace transform of $Y(s)$:

$$y(t) = -2 \sin t$$

d) $y'' = -y,\ y(0) = 1, y'(0) = 0$

Similarly as in case c):

$s^2\,Y(s) - s = -Y(s) \implies Y(s) = \frac{s}{s^2+1} \implies y(t) = \cos t$

e) $2\,y' - y = 0,\ y(0) = \frac{1}{2}$

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \implies \mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0)$

The Laplace transform of the differential equation is:

$$2\left(s\,Y(s) - \frac{1}{2}\right) - Y(s) = 0$$
$$2\,s\,Y(s) - 1 - Y(s) = 0$$
$$Y(s)\,(2\,s - 1) = 1$$
$$Y(s) = \frac{1}{2\,s-1} = \frac{1}{2}\cdot\frac{1}{s-\frac{1}{2}}$$

The solution of the differential equation is the inverse Laplace transform of $Y(s)$:

$$y(t) = \frac{1}{2}\,e^{\frac{1}{2}t}$$

f) $y' + 7\,y = 6,\ y(0) = 0$

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \implies \mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0)$

The Laplace transform of the differential equation is:

$$s\,Y(s) - 0 + 7\,Y(s) = \frac{6}{s}$$
$$(s+7)\,Y(s) = \frac{6}{s}$$
$$Y(s) = \frac{6}{s(s+7)} = \frac{A}{s} + \frac{B}{s+7} = \frac{A(s+7) + B\,s}{s(s+7)}$$

$$\implies \quad 6 = A(s+7) + B\,s$$
$$s = -7:\ \ 6 = 0 + B(-7) \implies B = -\frac{6}{7}$$
$$s = 0:\ \ 6 = A\cdot 7 + 0 \implies A = \frac{6}{7}$$
$$\implies Y(s) = \frac{6}{7}\cdot\frac{1}{s} - \frac{6}{7}\cdot\frac{1}{s+7}$$

The solution of the initial value problem is the inverse Laplace transform of $Y(s)$:

$$y(t) = \frac{6}{7} - \frac{6}{7}\,e^{-7\,t}$$

g) $2y' + y = e^{2t},\ y(0) = 1$

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \implies \mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0)$

The Laplace transform of the differential equation is:

$$2\,(s\,Y(s) - 1) + Y(s) = \frac{1}{s-2}$$
$$2\,s\,Y(s) - 2 + Y(s) = \frac{1}{s-2}$$
$$(2\,s+1)\,Y(s) = \frac{1}{s-2} + 2 = \frac{1 + 2\,(s-2)}{s-2} = \frac{2\,s-3}{s-2}$$

$$Y(s) = \frac{2\,s-3}{(2\,s+1)\,(s-2)} = \frac{A}{2\,s+1} + \frac{B}{s-2} = \frac{A(s-2) + B(2\,s+1)}{(2\,s+1)\,(s-2)}$$

$$\implies \quad 2\,s - 3 = A(s-2) + B(2\,s+1)$$
$$s = 2: \qquad 1 = 0 + B \cdot 5 \qquad \implies B = \frac{1}{5}$$
$$s = -\frac{1}{2}: \quad -4 = A \cdot \left(-\frac{5}{2}\right) + 0 \quad \implies A = \frac{8}{5}$$

$$\implies Y(s) = \frac{8}{5} \cdot \frac{1}{2\,s+1} + \frac{1}{5} \cdot \frac{1}{s-2} = \frac{8}{5} \cdot \frac{1}{2} \cdot \frac{1}{s+\frac{1}{2}} + \frac{1}{5} \cdot \frac{1}{s-2}$$

The solution of the initial value problem is:

$$y(t) = \frac{4}{5}\,e^{-\frac{1}{2}t} + \frac{1}{5}\,e^{2t}$$

h) $y'' + 3\,y' + 2\,y = e^{-t},\ y(0) = 0,\ y'(0) = 0$

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \implies \mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0),$

$\mathcal{L}\{y''(t)\}(s) = s^2\,Y(s) - s\,y(0) - y'(0)$

The Laplace transform of the differential equation is:

$$\left(s^2\,Y(s) - s \cdot 0 - 0\right) + 3\,(s\,Y(s) - 0) + 2\,Y(s) = \frac{1}{s+1}$$
$$\left(s^2 + 3\,s + 2\right)Y(s) = \frac{1}{s+1}$$
$$Y(s) = \frac{1}{\left(s^2 + 3\,s + 2\right)(s+1)} = \frac{1}{(s+2)\,(s+1)\,(s+1)} = \frac{1}{(s+2)\,(s+1)^2} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$\Longrightarrow \quad 1 = A(s+1)^2 + B(s+2)(s+1) + C(s+2)$$
$$s=-1:\quad 1 = 0+0+C\cdot 1 \quad \Longrightarrow C=1$$
$$s=-2:\quad 1 = A+0+0 \quad \Longrightarrow A=1$$
$$s=0:\quad 1 = A+B\cdot 2+C\cdot 2 \quad \Longrightarrow 2B = 1-1-2 \Longrightarrow B=-1$$

$$\Longrightarrow Y(s) = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

The solution of the initial value problem is:

$$y(t) = e^{-2t} - e^{-t} + t\,e^{-t}$$

i) $y'' + 2y' + 5y = 0,\ y(0) = 1,\ y'(0) = 0$

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \Longrightarrow \mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0),$
$\mathcal{L}\{y''(t)\}(s) = s^2\,Y(s) - s\,y(0) - y'(0)$

The Laplace transform of the differential equation is:

$$\left(s^2\,Y(s) - s\cdot 1 - 0\right) + 2\,(s\,Y(s) - 1) + 5\,Y(s) = 0$$
$$\left(s^2 + 2s + 5\right)Y(s) - s - 2 = 0$$
$$Y(s) = \frac{s+2}{s^2+2s+5} = \frac{s+2}{\mathbf{(s+1)^2+4}} = \frac{s+1}{(s+1)^2+4} + \frac{1}{2}\cdot\frac{2}{(s+1)^2+4}$$

The denominator has complex roots $\Longrightarrow$ completing the square

The solution of the initial value problem is:

$$y(t) = e^{-t}\cos(2t) + \frac{1}{2}\,e^{-t}\sin(2t)$$

j) $y' + y = \sin 3t,\ y(0) = 0$

Let $\mathcal{L}\{y(t)\}(s) = Y(s) \Longrightarrow \mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0)$

The Laplace transform of the differential equation is:

$$(s\,Y(s) - 0) + Y(s) = \frac{3}{s^2+9}$$
$$(s+1)\,Y(s) = \frac{3}{s^2+9}$$
$$Y(s) = \frac{3}{\mathbf{(s+1)\left(s^2+9\right)}} = \frac{\mathbf{A}}{\mathbf{s+1}} + \frac{\mathbf{B\,s+C}}{\mathbf{s^2+9}} = \frac{A\left(s^2+9\right) + (B\,s+C)(s+1)}{(s+1)\left(s^2+9\right)}$$

The roots of the denominator are $\mathbf{s_1 = -1}$ and $\mathbf{s_{2,3} = \pm 3\,i}$.

$$\Longrightarrow \quad 3 = A(s^2+9) + (B\,s + C)(s+1)$$

$s=-1$: $3 = A\cdot 10 + 0 \quad \Longrightarrow A = \frac{3}{10}$

$s=0$: $3 = A\cdot 9 + C\cdot 1 \quad \Longrightarrow C = 3 - 9A = 3 - \frac{27}{10} = \frac{3}{10}$

$s=1$: $3 = A\cdot 10 + 2B + 2C \Longrightarrow 2B = 3 - 10A - 2C = 3 - 3 - \frac{3}{5} \Longrightarrow B = -\frac{3}{10}$

$$\Longrightarrow Y(s) = \frac{3}{10}\cdot\frac{1}{s+1} + \frac{3}{10}\cdot\frac{-s+1}{s^2+9} = \frac{3}{10}\cdot\frac{1}{s+1} + \frac{3}{10}\left(-\frac{s}{s^2+9} + \frac{1}{3}\cdot\frac{3}{s^2+9}\right)$$

The solution of the initial value problem is:

$$y(t) = \frac{3}{10}\,e^{-t} - \frac{3}{10}\cos(3\,t) + \frac{1}{10}\sin(3\,t)$$

4.* Solving linear systems

A useful tool for the solution: Cramer’s rule:

$a\,x + b\,y = \boldsymbol{e}$
$c\,x + d\,y = \boldsymbol{f}$

If $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,d - b\,c \neq 0$ then the solution is unique and

$$x = \frac{\det\begin{pmatrix} \boldsymbol{e} & b \\ \boldsymbol{f} & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \quad \text{and} \quad y = \frac{\det\begin{pmatrix} a & \boldsymbol{e} \\ c & \boldsymbol{f} \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

Solve the following initial value problems:

a) $x' = 2\,x + y \qquad x(0) = 0$
$y' = -y \qquad y(0) = -3$

Let $\mathcal{L}\{x(t)\}(s) = X(s)$ and $\mathcal{L}\{y(t)\}(s) = Y(s)$
$\Longrightarrow \mathcal{L}\{x'(t)\}(s) = s\,X(s) - x(0)$ and $\mathcal{L}\{y'(t)\}(s) = s\,Y(s) - y(0)$

The Laplace transform of the differential equation system:

$$\Longrightarrow \begin{aligned} s\,X(s) - 0 &= 2\,X(s) + Y(s) \\ s\,Y(s) + 3 &= -Y(s) \end{aligned} \quad \text{or} \quad \begin{aligned} s\,X &= 2\,X + Y \\ s\,Y + 3 &= -Y \end{aligned}$$

$$\Longrightarrow \begin{aligned} (s-2)\,X - Y &= 0 \\ (s+1)\,Y &= -3 \end{aligned}$$

Applying the Cramer's rule:

$$X = \frac{\det\begin{pmatrix} 0 & -1 \\ -3 & s+1 \end{pmatrix}}{\det\begin{pmatrix} s-2 & -1 \\ 0 & s+1 \end{pmatrix}} = \frac{0\,(s+1)-(-1)\,(-3)}{(s-2)\,(s+1)-(-1)\cdot 0} = \frac{-3}{(s-2)\,(s+1)}$$

Partial fraction decomposition: $\dfrac{-3}{(s-2)\,(s+1)} = \dfrac{A}{s-2} + \dfrac{B}{s+1}$

$\Longrightarrow \quad -3 = A(s+1) + B(s-2)$

$s = -1$: $\quad -3 = 0 - 3B \Longrightarrow B = 1$

$s = 2$: $\quad -3 = 3A + 0 \Longrightarrow A = -1$

$$\Longrightarrow \quad X = \frac{1}{s+1} - \frac{1}{s-2} \quad \Longrightarrow \quad \mathbf{x(t) = e^{-t} - e^{2t}}$$

$$Y = \frac{\det\begin{pmatrix} s-2 & 0 \\ 0 & -3 \end{pmatrix}}{\det\begin{pmatrix} s-2 & -1 \\ 0 & s+1 \end{pmatrix}} = \frac{-3\,(s-2)}{(s-2)\,(s+1)} = -\frac{3}{s+1} \quad \Longrightarrow \quad \mathbf{y(t) = -3\,e^{-t}}$$

b) $x' = x + 3y \qquad x(0) = 1$

$y' = -x + 5y \qquad y(0) = 0$

$$\begin{aligned} s\,X(s) - 1 &= X(s) + 3\,Y(s) \\ s\,Y(s) &= -X(s) + 5\,Y(s) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} X(s) &= -\frac{1}{2\,(-4+s)} + \frac{3}{2\,(-2+s)} \\ Y(s) &= -\frac{1}{2\,(-4+s)} + \frac{1}{2\,(-2+s)} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x(t) &= \frac{3}{2}\,e^{2t} - \frac{1}{2}\,e^{4t} \\ y(t) &= \frac{1}{2}\,e^{2t} - \frac{1}{2}\,e^{4t} \end{aligned}$$

c) $x' = -8y \qquad x(0) = 1$

$y' = 2x \qquad y(0) = -2$

$$\begin{aligned} s\,X(s) - 1 &= -8\,Y(s) \\ s\,Y(s) + 2 &= 2\,X(s) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} X(s) &= \frac{s+16}{s^2+16} \\ Y(s) &= \frac{-2\,s+2}{s^2+16} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x(t) &= \cos 4\,t + 4 \sin 4\,t \\ y(t) &= -2\cos 4\,t + \frac{1}{2}\sin 4\,t \end{aligned}$$

d) $x' = 3x - 2y \qquad x(0) = 1$

$y' = 2x + 5y \qquad y(0) = 1$

$$\begin{aligned} s\,X(s) - 1 &= 3\,X(s) - 2\,Y(s) \\ s\,Y(s) - 1 &= 2\,X(s) + 5\,Y(s) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} X(s) &= \frac{-7+s}{19-8\,s+s^2} \\ Y(s) &= \frac{-1+s}{19-8\,s+s^2} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x(t) &= e^{4t}\left(\cos\left(\sqrt{3}\,t\right) - \sqrt{3}\,\sin\left(\sqrt{3}\,t\right)\right) \\ y(t) &= e^{4t}\left(\cos\left(\sqrt{3}\,t\right) + \sqrt{3}\,\sin\left(\sqrt{3}\,t\right)\right) \end{aligned}$$

e) $x' = 3x \qquad x(0) = 4$

$y' = x + 3y \qquad y(0) = 2$

$$s\,X(s) - 4 = 3\,X(s) \quad \Longrightarrow \quad X(s) = \frac{4}{-3+s} \quad \Longrightarrow \quad x(t) = 4\,e^{3\,t}$$

$$s\,Y(s) - 2 = X(s) + 3\,Y(s) \qquad Y(s) = \frac{4}{(-3+s)^2} + \frac{2}{-3+s} \qquad y(t) = 2\,e^{3\,t} + 4\,t\,e^{3\,t}$$

f) $x' = 3\,x + y \qquad x(0) = 4$

$y' = -x + y \qquad y(0) = 2$

$$s\,X(s) - 4 = 3\,X(s) + Y(s) \quad \Longrightarrow \quad X(s) = \frac{6}{(-2+s)^2} + \frac{4}{-2+s} \quad \Longrightarrow \quad x(t) = 4\,e^{2\,t} + 6\,t\,e^{2\,t}$$

$$s\,Y(s) - 2 = -X(s) + Y(s) \qquad Y(s) = -\frac{6}{(-2+s)^2} + \frac{2}{-2+s} \qquad y(t) = 2\,e^{2\,t} + 6\,t\,e^{2\,t}$$

g) $x' = 4\,y + 1 \qquad x(0) = 1$

$y' = x + t \qquad y(0) = 0$

$$s\,X(s) - 1 = 4\,Y(s) + \frac{1}{s} \quad \Longrightarrow \quad X(s) = \frac{1}{-2+s} - \frac{1}{s^2} \quad \Longrightarrow \quad x\,(t) = e^{2\,t} - t$$

$$s\,Y(s) = X(s) + \frac{1}{s^2} \qquad Y(s) = \frac{1}{2\,(-2+s)} - \frac{1}{2\,s} \qquad y(t) = -\frac{1}{2} + \frac{1}{2}\,e^{2\,t}$$

h) $x' = x + 3\,y + 8 \qquad x(0) = 0$

$y' = x - y \qquad y(0) = 0$

$$s\,X(s) = X(s) + 3\,Y(s) + \frac{8}{s} \quad \Longrightarrow \quad X(s) = \frac{3}{-2+s} - \frac{2}{s} - \frac{1}{2+s} \quad \Longrightarrow \quad x(t) = -2 - e^{-2\,t} + 3\,e^{2\,t}$$

$$s\,Y(s) = X(s) - Y(s) \qquad Y(s) = \frac{1}{-2+s} - \frac{2}{s} + \frac{1}{2+s} \qquad y(t) = -2 + e^{-2\,t} + e^{2\,t}$$