04 - Higher order linear differential equations

Definition. An nth order linear differential equation has the form

(1) $y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots + a_1(x)\, y' + a_0(x)\, y = f(x)$

(2) $f(x) \equiv 0$: homogeneous equation
(3) $f(x) \not\equiv 0$: nonhomogeneous equation

Initial values:

(4) $y(x_0) = y_0,\ y'(x_0) = y_{0,1},\ \ldots,\ y^{(n-1)}(x_0) = y_{0,n-1}$

Theorem: If the functions $f,\ a_0,\ a_1,\ \ldots,\ a_{n-1}$ are all continuous on some open interval I containing x_0 then the initial value problem given by (1), (4) has a unique solution on the interval I.

Theorem: If y_1 and y_2 are solutions of the nonhomogeneous equation (3) then $y_1 - y_2$ is a solution of the homogeneous equation (2).

Proof: Let $L[y] := y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots + a_1(x)\, y' + a_0(x)\, y$
Since $L[y_1] = f(x),\ L[y_2] = f(x)$ and
$(y_1 - y_2)' = y_1' - y_2',\ (y_1 - y_2)'' = y_1'' - y_2''$, etc.
$\Longrightarrow L[y_1 - y_2] = L[y_1] - L[y_2] = f(x) - f(x) = 0.$

Theorem (consequence): The general solution of the nonhomogeneous equation (3) on I is $\boldsymbol{y(x) = y_h(x) + y_p(x)}$ where $y_h(x)$ is the general solution of the homogeneous equation (2) and $y_p(x)$ is a particular solution of (3).

Proof: If y is any solution of (3) and y_p is a particular solution of (3) then $y - y_p$ is a solution of (2). Let $y - y_p = y_h \Longrightarrow y = y_p + y_h$, where y_h is the general solution of (2).

Theorem (Superposition principle or linearity principle):
For the homogeneous linear differential equation (2), sums and constant multiples of solutions on some open interval I are also solutions of (2) on I.

Proof: Let $L[y] := y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots + a_1(x)\, y' + a_0(x)\, y$
We show that if $L[Y_1] \equiv 0$ and $L[Y_2] \equiv 0$ then $L[Y_1 + Y_2] \equiv 0$.

$Y_1^{(n)} + a_{n-1}(x)\, Y_1^{(n-1)} + \ldots + a_1(x)\, Y_1' + a_0(x)\, Y_1 \equiv 0$
$Y_2^{(n)} + a_{n-1}(x)\, Y_2^{(n-1)} + \ldots + a_1(x)\, Y_2' + a_0(x)\, Y_2 \equiv 0$

$\Longrightarrow (Y_1 + Y_2)^{(n)} + a_{n-1}(x)\, (Y_1 + Y_2)^{(n-1)} + \ldots + a_1(x)\, (Y_1 + Y_2)' + a_0(x)\, (Y_1 + Y_2) \equiv 0$
$\Longrightarrow L[Y_1 + Y_2] \equiv 0$

Similarly, if $L[Y_1] \equiv 0$ then $L[C \cdot Y_1] = 0$.

Warning: The theorem does not hold for the nonhomogeneous equation or for nonlinear equations.

Linear independence of solutions

Definition: n solutions $y_1, y_2, \ldots, y_n$ form a **basis (or fundamental system) of solutions** of (2) on I if these solutions are linearly independent on I.

Theorem: If $y_1, y_2, \ldots, y_n$ are linearly independent solutions of the homogeneous equation (2) then the general solution is of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x), \quad c_i \in \mathbb{R}$$

Definition: The functions $y_1, y_2, \ldots, y_n$ are called **linearly independent** on an interval I if the equation

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + \ldots + \lambda_n y_n(x) = 0, \text{ for all } x \in I$$

implies that all $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.
These functions are called **linearly dependent** on I if this equation holds on I for some $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero.

Remarks: (1) If for example $\lambda_1 \neq 0$ then $y_1(x)$ can be expressed as the following linear combination:

$$y_1(x) = -\frac{1}{\lambda_1}(\lambda_2 y_2(x) + \ldots + \lambda_n y_n(x))$$

(2) If $n = 2$ and $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$ then $y_1 = -\frac{\lambda_2}{\lambda_1} y_2$ or $y_2 = -\frac{\lambda_1}{\lambda_2} y_1$.

$\Longrightarrow$ y_1 and y_2 are proportional, while in the case of linear independence they are not proportional.

Examples: (1) The functions $y_1 = x,\ y_2 = 2x - x^2,\ y_3 = 3x^2$ are linearly dependent on any interval since $2y_1 - y_2 - \frac{1}{3} y_3 = 0$.

(2) The functions $y_1 = x,\ y_2 = x^2,\ y_3 = x^3$ are linearly independent on any interval since if $\lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = 0$ for all $x \in \mathbb{R}$ then with $x = -1,\ 1,\ 2$, we get

$$-\lambda_1 + \lambda_2 - \lambda_3 = 0$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$2\lambda_1 + 4\lambda_2 + 8\lambda_3 = 0$$

from where $\lambda_1 = \lambda_2 = \lambda_3 = 0$, that is, $y_1,\ y_2$ and y_3 are linearly independent.

Definition: Let $y_1, y_2, \ldots, y_n$ be at least $(n-1)$ times continuously differentiable on I.
The Wronski determinant or **Wronskian** of $y_1, y_2, \ldots, y_n$ is

$$W(x) = \det\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}$$

Theorem: (1) If $W \not\equiv 0$ on I then $y_1, y_2, \dots, y_n$ are linearly independent on I.

Remark: The converse of statement (1) is not true. For example, $y_1(x) = x^3$ and $y_2(x) = |x|^3$ are linearly independent on I but

$$W = \det\begin{pmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{pmatrix} \equiv 0 \text{ if } x \geq 0 \quad \text{and} \quad W = \det\begin{pmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{pmatrix} \equiv 0 \text{ if } x \leq 0.$$

(2) If $y_1, y_2, \dots, y_n$ are linearly dependent on I then $W \equiv 0$.

Examples: (1) The functions $y_1 = x,\ y_2 = 2x - x^2,\ y_3 = 3x^2$ are linearly dependent so

$$W = \det\begin{pmatrix} x & 2x - x^2 & 3x^2 \\ 1 & 2 - 2x & 6x \\ 0 & -2 & 6 \end{pmatrix} = x(12 - 12x + 12x) - 1 \cdot (12x - 6x^2 + 6x^2) = 12x - 12x = 0.$$

(2) If $y_1 = x,\ y_2 = x^2,\ y_3 = x^3$ then

$$W = \det\begin{pmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{pmatrix} = x(12x^2 - 6x^2) - 1 \cdot (6x^3 - 2x^3) \equiv 6x^3 - 4x^3 = 2x^3 \neq 0$$

so y_1, y_2, y_3 are linearly independent.

Theorem: Suppose that the coefficients $a_0(x),\ a_1(x),\ \dots,\ a_{n-1}(x)$ of the homogeneous equation (2) are continuous on some open interval I. Then n solutions $y_1, y_2, \dots, y_n$ of (2) on I are linearly independent if and only if $W(x) \neq 0$ if $x \in I$.

Higher order homogeneous equations with constant coefficients

Consider the equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0, \quad \text{where } a_i \in \mathbb{R},\ i = 0, 1, \dots, n-1$$

We assume that the equation has a solution of the form $y = e^{\lambda x}$ (recall that a first order linear differential equation $y' + K y = 0$ has a solution $y = e^{-K x}$, where K is a constant).

Substituting $y = e^{\lambda x},\ \lambda \in \mathbb{R}:\ y' = \lambda e^{\lambda x},\ y'' = \lambda^2 e^{\lambda x},\ \dots,\ y^{(n)} = \lambda^n e^{\lambda x}$

$$\Longrightarrow e^{\lambda x}(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0) = 0, \quad e^{\lambda x} \neq 0$$

We obtain the **characteristic equation**:

$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$

Cases:

1. Distinct real roots: If all the n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real and different then the solutions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x}$ are linearly independent (they constitute a basis).

Example: $y_1 = e^{2x},\ y_2 = e^{3x}$

$$\Longrightarrow W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{pmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0$$

$\Longrightarrow y_1 = e^{2x},\ y_2 = e^{3x}$ are linearly independent.

2. Multiple real roots: If λ is a root of order k, then k linearly independent solutions corresponding to this root are $e^{\lambda x}, x e^{\lambda x}, x^2 e^{\lambda x}, \ldots, x^{k-1} e^{\lambda x}$ (it is called **inner resonance**).

3. Simple complex roots: If $\lambda_{1,2} = \alpha \pm \beta i$ then the solutions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are linearly independent.

If complex roots occur, they must occur in conjugate pairs since the coefficients of the characteristic equation are real. The previous statements are also true for complex roots, however, in this case the solutions are complex. In the following, we use the **Euler's formula** $e^{ix} = \cos x + i\sin x$.

If the roots are $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \overline{\lambda_1} = \alpha - \beta i$ then two complex solutions are

$$Y_1 = e^{\lambda_1 x} = e^{(\alpha+\beta i)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$
$$Y_2 = e^{\lambda_2 x} = e^{(\alpha-\beta i)x} = e^{\alpha x} e^{i(-\beta)x} = e^{\alpha x}(\cos(-\beta x) + i\sin(-\beta x)) = e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

Any linear combination of Y_1 and Y_2 is also a solution $\Longrightarrow$

$$Z_1 := \frac{Y_1 + Y_2}{2} = e^{\alpha x}\cos\beta x = \mathrm{Re}\left(e^{\lambda_1 x}\right) \quad \text{(this is the real part of } e^{\lambda_1 x}\text{)}$$
$$Z_2 := \frac{Y_1 - Y_2}{2i} = e^{\alpha x}\sin\beta x = \mathrm{Im}\left(e^{\lambda_1 x}\right) \quad \text{(this is the imaginary part of } e^{\lambda_1 x}\text{)}$$

Z_1 and Z_2 are linearly independent (since they are not a constant multiple of each other). We use Z_1 and Z_2 instead of Y_1 and Y_2.

4. Multiple complex roots: Z_1 and Z_2 are multiplied by x, x^2, x^3, etc.

Examples

1. $y''' - 2y'' - 3y' = 0$

The characteristic equation is $\lambda^3 - 2\lambda^2 - 3\lambda = 0$

$\lambda(\lambda + 1)(\lambda - 3) = 0 \Longrightarrow \lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = 3$ (distinct real roots)

The linearly independent solutions are $e^{0x} = 1,\ e^{-x},\ e^{3x}$
The general solution of the homogeneous equation is a linear combination of these functions:
$y_h(x) = c_1 + c_2\, e^{-x} + c_3\, e^{3x}$ where $c_1,\ c_2,\ c_3 \in \mathbb{R}$

2. $y''' + 2\,y'' + y' = 0$

The characteristic equation is $\lambda^3 + 2\,\lambda^2 + \lambda = \lambda\left(\lambda^2 + 2\,\lambda + 1\right) = \lambda(\lambda + 1)^2 = 0$
$\Longrightarrow\ \lambda_1 = 0,\ \lambda_2 = \lambda_3 = -1$ (double real roots, inner resonance)

The linearly independent solutions are $e^{0x} = 1,\ e^{-x},\ x\,e^{-x}$
The general solution is: $y_h(x) = c_1 + c_2\, e^{-x} + c_3\, x\, e^{-x}$

3. $y''' + 4\,y'' + 13\,y' = 0$

The characteristic equation is $\lambda^3 + 4\,\lambda^2 + 13\,\lambda = \lambda\left(\lambda^2 + 4\,\lambda + 13\right) = 0$
$\Longrightarrow\ \lambda_1 = 0,\ \lambda_{2,3} = \dfrac{-4 \pm \sqrt{16 - 52}}{2} = \dfrac{-4 \pm 6\,i}{2} = -2 \pm 3\,i$ (simple complex roots)

The linearly independent solutions are $e^{0x} = 1,\ e^{-2x}\cos 3\,x,\ e^{-2x}\sin 3\,x$
The general solution is: $y_h(x) = c_1 + c_2\, e^{-2x}\cos 3\,x + c_3\, e^{-2x}\sin 3\,x$

4. $y^{(4)} + 2\,y'' + y = 0$

The characteristic equation is $\lambda^4 + 2\,\lambda^2 + 1 = \left(\lambda^2 + 1\right)^2 = (\lambda - i)^2\,(\lambda + i)^2 = 0$
$\Longrightarrow\ \lambda_1 = \lambda_2 = i,\ \lambda_3 = \lambda_4 = -i$ (double complex roots, inner resonance)

The linearly independent solutions are $e^{0x}\cos x = \cos x,\ \sin x,\ x\cos x,\ x\sin x$
The general solution is: $y_h(x) = c_1\cos x + c_2\sin x + c_3\, x\cos x + c_4\, x\sin x$

Higher order linear nonhomogeneous differential equations

Higher order nonhomogeneous equations with constant coefficients

Consider the equation

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \ldots + a_1\, y' + a_0\, y = f(x)$$

with constant coefficients where $f(x) \neq 0$. The general solution of the nonhomogeneous linear equation is

$$y = y_h + y_p$$

where y_h is the general solution of the corresponding homogeneous equation and y_p is a particular solution of the nonhomogeneous equation.

If $f(x)$ has a special form: **exponential function, polynomial, cosine, sine or sums or products of such functions** then the derivatives of $f(x)$ are similar to $f(x)$ itself.
In these cases we can find y_p with the method of undetermined coefficients:
we suppose that the form of y_p is similar to the form of $f(x)$.

Rules for the method of undetermined coefficients

1. Basic Rule. If $f(x)$ has the form

1. $f(x) = K\, e^{\alpha x}$
2. $f(x) = P_m(x) = a_m x^m + \ldots + a_0$
3. $f(x) = K \sin \beta x$ or $K \cos \beta x$
4. $f(x) = K\, e^{\alpha x} \sin \beta x$ or $K\, e^{\alpha x} \cos \beta x$
5. $f(x) = P_m(x) \sin \beta x$ or $P_m(x) \cos \beta x$, where $P_m(x) = a_m x^m + \ldots + a_0$
6. $f(x) = P_m(x)\, e^{\alpha x}$, where $P_m(x) = a_m x^m + \ldots + a_0$
7. $f(x) = P_m(x)\, e^{\alpha x} \sin \beta x$ or $P_m(x)\, e^{\alpha x} \cos \beta x$, where $P_m(x) = a_m x^m + \ldots + a_0$

then the choice for y_p is

1. $y_p = A\, e^{\alpha x}$, where A is unknown
2. $y_p = Q_m(x) = B_m x^m + \ldots + B_0$, where $B_0, \ldots, B_m$ are unknowns
3. $y_p = A \sin \beta x + B \cos \beta x$, where A, B are unknowns
4. $y_p = e^{\alpha x}(A \sin \beta x + B \cos \beta x)$, where A, B are unknowns
5. $y_p = Q_m(x) \sin \beta x + R_m(x) \cos \beta x$, where $Q_m(x)$, $R_m(x)$ are unknown polynomials of degree m
6. $y_p = Q_m(x)\, e^{\alpha x}$, where $Q_m(x)$ is an unknown polynomial of degree m
7. $y_p = e^{\alpha x}(Q_m(x) \sin \beta x + R_m(x) \cos \beta x)$, where$Q_m(x)$, $R_m(x)$ are unknown polynomials of degree m

The unknown coefficients in y_p can be determined by substituting y_p and its derivatives into the nonhomogeneous equation and comparing the corresponding terms on both sides.
If the choice for y_p is correct then we get the same number of independent linear equations as the number of unknowns, so the solution for the unknowns is unique.

2. Sum Rule. If $f(x)$ is a sum of functions in the above list then y_p is also the sum of the corresponding functions.

3. Modification Rule The method doesn't work if a term in $f(x)$ (or y_p) happens to be a solution of the homogeneous equation. It is called **outer resonance**, and this term is multiplied by x, x^2, ... etc. until the resonance disappears.

Practice exercises - Homework

Solve the following differential equations.

1. $y'' - 3y' + 2y = (e^{3x}) + (x^2 + x)$

Solution. The general solution of the homogeneous equation:

$$\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0 \implies y_h = C_1 e^{2x} + C_2 e^x$$

Here $f(x) = (e^{3x}) + (x^2 + x)$. Finding a particular solution of the nonhomogeneous equation:

$$\begin{aligned} 2\cdot &\mid\ y_p := (A e^{3x}) + (B x^2 + C x + D) \\ -3\cdot &\mid\ y_p{}' = 3 A e^{3x} + 2 B x + C \\ 1\cdot &\mid\ y_p{}'' = 9 A e^{3x} + 2 B \end{aligned}$$

Substituting into the nonhomogeneous equation:

$$(9A - 9A + 2A) e^{3x} + x^2(2B) + x(2C - 6B) + (2D - 3C + 2B) = e^{3x} + x^2 + x$$

Comparing the coefficients of the corresponding terms on both sides:

$$2A = 1 \implies A = \frac{1}{2}$$
$$2B = 1 \implies B = \frac{1}{2}$$
$$2C - 6B = 1 \implies 2C = 4,\ C = 2$$
$$2D - 3C + 2B = 0 \implies 2D = 6 - 1,\ D = \frac{5}{2}$$

The general solution of the nonhomogeneous equation is

$$y(x) = y_h(x) + y_p(x) = C_1 e^{2x} + C_2 e^x + \frac{1}{2} x^2 + 2x + \frac{5}{2} + \frac{1}{2} e^{3x}$$

2. $y'' - 3y' + 2y = (x) + (e^x)$

Solution. The general solution of the homogeneous equation:

$$\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0 \implies y_h = C_1 e^{2x} + \boldsymbol{C_2 e^x}$$

Here $f(x) = (x) + (\boldsymbol{e^x})$. Now we have outer resonance. Let's see what happens if we don't observe this and
make a wrong choice for y_p. Based on the structure of $f(x)$ only:

$$\begin{aligned} 2\cdot &\mid\ y_p := (A x + B) + (C e^x) \\ -3\cdot &\mid\ y_p{}' = A + C e^x \\ 1\cdot &\mid\ y_p{}'' = C e^x \end{aligned}$$

Substituting into the nonhomogeneous equation:

$x(2A) + (2B - 3A) + (2C - 3C + C)e^x = x + e^x$

This is a contradiction since the coefficient of e^x is 0 on the left-hand side and 1 on the right-hand side and $0 \neq 1$.

Since the term $C e^x$ in y_p (or e^x in $f(x)$) is a constant multiple of the term $C_2 e^x$ in y_h then we multiply $C e^x$ by x, so the right choice for y_p is the following:

$$\begin{array}{r|l} 2\cdot & y_p := (Ax + B) + (\mathbf{C\,x\,e^x}) \quad \Longleftarrow y_h = C_1 e^{2x} + \mathbf{C_2\,e^x} \\ -3\cdot & y_p{}' = A + C x e^x + C e^x \\ 1\cdot & y_p{}'' = C x e^x + C e^x + C e^x \end{array}$$

Substituting into the nonhomogeneous equation and comparing the coefficients on both sides:

$x(2A) + (2B - 3A) + x e^x(2C - 3C + C) + e^x(-3C + 2C) = x + e^x$

$$2A = 1 \implies A = \frac{1}{2}$$
$$2B - 3A = 0 \implies B = \frac{3}{2}A = \frac{3}{4}$$
$$-C = 1 \implies C = -1$$

The general solution of the nonhomogeneous equation is

$$y(x) = y_h(x) + y_p(x) = C_1 e^{2x} + C_2 e^x + \frac{1}{2}x + \frac{3}{4} - x e^x$$

3. $y'' - y = (x^2 - x + 1) + (e^x)$

Solution. The general solution of the homogeneous equation:

$$\lambda^2 - 1 = 0 \implies \lambda_1 = 1,\ \lambda_2 = -1 \quad \implies y_h = \mathbf{C_1\,e^x} + C_2 e^{-x}$$

Here $f(x) = (x^2 - x + 1) + (\mathbf{e^x})$, so there is outer resonance.

$$\begin{array}{r|l} -1\cdot & y_p := (A x^2 + B x + C) + (\mathbf{D\,x\,e^x}) \\ 0\cdot & y_p{}' = 2Ax + B + D x e^x + D e^x \\ 1\cdot & y_p{}'' = 2A + D x e^x + D e^x + D e^x \end{array}$$

Substituting into the nonhomogeneous equation and comparing the coefficients on both sides:

$(-A)x^2 + (-B)x + (2A - C) + x e^x(-D + D) + e^x \cdot 2D = x^2 - x + 1 + e^x$

$A = -1, B = 1,\ C = 2A - 1 = -3,\ D = \frac{1}{2}$

The general solution of the nonhomogeneous equation is

$y(x) = y_h(x) + y_p(x) = C_1 e^x + C_2 e^{-x} - x^2 + x - 3 + \frac{1}{2} x e^x$

4. $y'' - 2y' + y = 6e^x$

Solution. The general solution of the homogeneous equation:

$\lambda^2 - 2\lambda + 1 = 0 \implies \lambda_1 = \lambda_2 = 1 \implies y_h = \boldsymbol{C_1 e^x} + \boldsymbol{C_2 x e^x}$ (inner resonance)

First try: $y_p = \boldsymbol{A e^x} \implies$ substituting into the equation: $e^x(A - 2A + A) = 6e^x$, that is, $0 = 6$, which is a contradiction. This choice is not correct since $A e^x$ is a constant multiple of $C_1 e^x$ in y_h $\implies$ we have to multiply this term by x

Second try: $y_p = \boldsymbol{A x e^x}$. This choice is not correct either since $A x e^x$ is a constant multiple of the term $C_2 x e^x$ in y_h.
$\implies$ we multiply this term again by x

So the correct choice for y_p is the following:

$$\begin{array}{r|l} 1\cdot & y_p := \boldsymbol{A x^2 e^x} \quad \text{(outer resonance)} \\ -2\cdot & y_p' = \boldsymbol{2Axe^x} + \boldsymbol{Ax^2 e^x} \\ 1\cdot & y_p'' = \boldsymbol{2Ae^x} + \boldsymbol{2Axe^x} + \boldsymbol{2Axe^x} + \boldsymbol{Ax^2e^x} \end{array}$$

$\boldsymbol{x^2 e^x(A - 2A + A)} + \boldsymbol{x e^x(-4A + 4A)} + \boldsymbol{2Ae^x} = 6e^x$

$2A = 6 \implies A = 3$

The general solution of the nonhomogeneous equation is

$y(x) = y_h(x) + y_p(x) = C_1 e^x + C_2 x e^x + 3x^2 e^x$

5. $y'' + 8y' + 25y = e^{-4x}$

Solution. The general solution of the homogeneous equation:

$\lambda^2 + 8\lambda + 25 = 0 \implies \lambda_{1,2} = \frac{-8 \pm \sqrt{64 - 100}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i$

$y_h(x) = C_1 e^{-4x} \cos 3x + C_2 e^{-4x} \sin 3x$

Here $f(x) = e^{-4x}$. There is no outer resonance in this case, since e^{-4x} is not a constant multiple of either of the terms in y_h.

$$
\begin{aligned}
25\cdot &\mid\; y_p := A e^{-4x} \\
8\cdot &\mid\; y_p' = -4 A e^{-4x} \\
1\cdot &\mid\; y_p'' = 16 A e^{-4x}
\end{aligned}
$$

$$(25A - 32A + 16A)\, e^{-4x} = e^{-4x}$$

$$9A = 1 \implies A = \frac{1}{9}$$

The general solution of the nonhomogeneous equation is

$$y(x) = y_h(x) + y_p(x) = C_1 e^{-4x} \cos 3x + C_2 e^{-4x} \sin 3x + \frac{1}{9} e^{-4x}$$

6. $y'' + 5y' + 6y = 2e^{-2x}$, $y(0) = 0$, $y'(0) = 3$

Solution. The general solution of the homogeneous equation:

$\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0 \implies \lambda_1 = -2, \lambda_2 = -3$

$y_h(x) = \boldsymbol{C_1 e^{-2x}} + C_2 e^{-3x}$

Here $f(x) = \boldsymbol{2e^{-2x}}$. There is outer resonance, so the choice $y_p = \boldsymbol{A e^{-2x}}$ is not correct, since it is a constant multiple of $\boldsymbol{C_1 e^{-2x}}$ in $y_h(x)$. So the correct choice for y_p is:

$$
\begin{aligned}
6\cdot &\mid\; y_p := A x e^{-2x} \\
5\cdot &\mid\; y_p' = A e^{-2x} - 2 A x e^{-2x} \\
1\cdot &\mid\; y_p'' = -2 A e^{-2x} - 2 A e^{-2x} + 4 A x e^{-2x}
\end{aligned}
$$

$$x e^{-2x}(6A - 10A + 4A) + e^{-2x}(5A - 4A) = 2e^{-2x}$$

$$\implies A = 2$$

The general solution of the nonhomogeneous equation is

$$y(x) = y_h(x) + y_p(x) = C_1 e^{-2x} + C_2 e^{-3x} + 2x e^{-2x}$$

For the initial conditions we need y':

$$y'(x) = -2C_1 e^{-2x} - 3C_2 e^{-3x} + 2e^{-2x} - 4x e^{-2x}$$

From the initial conditions:

$y(0) = 0 \implies C_1 + C_2 = 0$

$y'(0) = 3 \implies -2C_1 - 3C_2 + 2 = 3 \implies C_1 = 1, C_2 = -1$

The solution of the initial value problem is

$$y(x) = e^{-2x} - e^{-3x} + 2x e^{-2x}$$

7. $y'' + y = (-4\cos x) + (x),\ y(0) = 2,\ y'(0) = 2$

Solution. $\lambda^2 + 1 = 0 \implies \lambda_{1,2} = \pm i \implies y_h(x) = \mathbf{C_1 \cos x} + \mathbf{C_2 \sin x}$

Here $f(x) = (-4\cos x) + (x)$
First try: $y_p = (\mathbf{A \cos x} + \mathbf{B \sin x}) + (C x + D)$ but this is not correct since $A\cos x$ is a constant multiple of $C_1 \cos x$ and $B \sin x$ is a constant multiple of $C_2 \sin x$ in $y_h(x)$.

There is outer resonance, so we multiply these two terms by x.

$$\begin{array}{r|l} 1\cdot & y_p := (\mathbf{A\,x \cos x} + \mathbf{B\,x \sin x}) + (C\,x + D) \\ 0\cdot & y_p' = A\cos x - A\,x\sin x + B\sin x + B\,x\cos x + C \\ 1\cdot & y_p'' = -A\sin x - A\sin x - A\,x\cos x + B\cos x + B\cos x - B\,x\sin x \end{array}$$

$(A - A)\,x\cos x + (B - B)\,x \sin x + (2\,B)\cos x + (-2\,A)\sin x + C\,x + D = -4\cos x + x$

$2\,B = -4,\ \ B = -2$

$-2\,A = 0,\ \ A = 0$

$C = 1,\ \ D = 0$

The general solution of the nonhomogeneous equation is
$y(x) = y_h(x) + y_p(x) = C_1\cos x + C_2 \sin x - 2\,x\sin x + x$

For the initial conditions we need y':
$y'(x) = -C_1 \sin\ + C_2 \cos x - 2\sin x - 2\,x\cos x + 1$

From the initial conditions:
$y(0) = 2 \implies C_1 = 2$
$y'(0) = 2 \implies C_2 + 1 = 2 \implies C_2 = 1$

The solution of the initial value problem is
$y(x) = 2\cos x + \sin x + x(1 - 2\sin x)$

First order linear differential equations with constant coefficients

1. Mixing problem: $y' = 0.6 - 0.2\,y$

See exercise 02-7. We can apply different solution methods:
(1) separable (autonomous): $\dfrac{dy}{dt} = 0.2\,(3 - y) \implies \displaystyle\int \frac{1}{3-y}\,dy = \int 0.2\,dt \ldots$
(2) first-order linear nonhomogeneous equation:

homogeneous equation: $y' = -0.2\,y \implies y_h(t) = C\,e^{-0.2\,t}$
variation of the constant method: $y_p(t) = c(t)\,e^{-0.2\,t} \implies$
$c'(t)\,e^{-0.2\,t} + c(t)\,e^{-0.2\,t}(-0.2) = 0.6 - 0.2\,c(t)\,e^{-0.2\,t} \implies c'(t) = 0.6\,e^{0.2\,t} \implies$
$c(t) = 3\,e^{0.2\,t} \implies y_p(t) = 3 \implies$
$y(t) = y_h(t) + y_p(t) = C\,e^{-0.2\,t} + 3$

(3) **first-order linear nonhomogeneous equation with constant coefficients:**

$y' + 0.2\,y = 0.6$
characteristic equation: $\lambda + 0.2 = 0 \implies y_h(t) = C\,e^{-0.2\,t}$
the particular solution of the nonhomogeneous equation: $y_p(t) = A \implies$
$0 + 0.2\,A = 0.6 \implies A = 3$
the general solution of the nonhomogeneous equation:
$y(t) = y_h(t) + y_p(t) = C\,e^{-0.2\,t} + 3$

2. Current in an RC circuit: $R\,I'(t) + \frac{1}{C}\,I(t) = F(t)$

See exercise 03-6. We solve it in the case when $R = C = 1$ and $F(t) = F_0 \sin t$ where $F_0 > 0$.
Homogeneous equation: $I'(t) + I(t) = 0$
Characteristic equation: $\lambda + 1 = 0 \implies \lambda = -1$
The general solution of the homogeneous equation: $I_h(t) = C\,e^{-t}$

If $F(t) = F_0 \sin t$ then the particular solution of the nonhomogeneous equation:

$1\cdot \mid I_p(t) = a \sin t + b \cos t$
$1\cdot \mid I_p'(t) = a \cos t - b \sin t$

Substituting into the nonhomogeneous equation:
$\sin t(a - b) + \cos t(b + a) = F_0 \sin t \implies$

$$\begin{array}{l} a - b = F_0 \\ b + a = 0 \end{array} \implies a = \frac{F_0}{2},\quad b = -\frac{F_0}{2}$$

The general solution of the nonhomogeneous equation:
$I(t) = I_h(t) + I_p(t) = C\,e^{-t} + \frac{F_0}{2}(\sin t - \cos t)$

Physical examples

1. Simple harmonic motion (mass on a spring)

Newton's second law: $m\,a = m\,x'' = -D\,x$

m: mass, a: acceleration, D: spring constant, x: displacement from the equilibrium position

$-D\,x$: spring force, $\omega = \sqrt{\frac{D}{m}}$: angular frequency

$\Longrightarrow$ $\mathbf{x''(t) + \omega^2\,x(t) = 0}$

Characteristic equation: $\lambda^2 + \omega^2 = 0 \Longrightarrow \lambda_{1,2} = \pm\,\omega\,i$

The general solution of the homogeneous equation: $\mathbf{x_h(t) = c_1 \sin \omega\,t + c_2 \cos \omega\,t}$

Remark: The equation can also be written in the form $x_h(t) = A\cos(\omega\,t - \alpha)$

where $A = \sqrt{{c_1}^2 + {c_2}^2}$ and $\tan\alpha = \frac{c_1}{c_2}$.

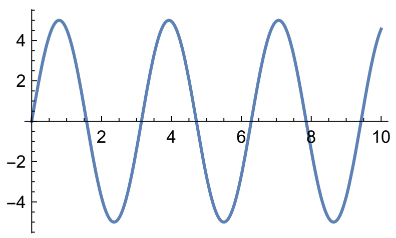
$$\tan\alpha = \frac{c_1}{c_2} \Longrightarrow \sin\alpha = \frac{c_1}{\sqrt{{c_1}^2 + {c_2}^2}},\ \cos\alpha = \frac{c_2}{\sqrt{{c_1}^2 + {c_2}^2}} \Longrightarrow$$

$$x_h(t) = c_1 \sin\omega\,t + c_2\cos\omega\,t = \sqrt{{c_1}^2 + {c_2}^2}\left(\frac{c_1}{\sqrt{{c_1}^2 + {c_2}^2}}\sin\omega\,t + \frac{c_2}{\sqrt{{c_1}^2 + {c_2}^2}}\cos\omega\,t\right) =$$

$$= \sqrt{{c_1}^2 + {c_2}^2}\,(\sin\alpha\cdot\sin\omega\,t + \cos\alpha\cdot\cos\omega\,t) = \sqrt{{c_1}^2 + {c_2}^2}\cdot\cos(\omega\,t - \alpha).$$

Example: $x''(t) + 4\,x(t) = 0,\ \ x(0) = 0,\ \ x'(0) = 10$

$\Longrightarrow x(t) = 5\sin(2\,t)$



2. Damped harmonic motion

Newton's second law: $m\,a = -D\,x - c\,v$

$-D\,x$: spring force, $-c\,v$: linear damping force

Let $\omega^2 = \frac{D}{m}$ and $2\,k = \frac{c}{m}$

$\Longrightarrow$ $\boldsymbol{x''(t) + 2\,k\,x'(t) + \omega^2\,x(t) = 0}$

Characteristic equation: $\lambda^2 + 2\,k\,\lambda + \omega^2 = 0 \Longrightarrow \lambda_{1,2} = -k \pm \sqrt{k^2 - \omega^2}$

a) $k > \omega$ (overdamping - distinct real roots)

b) $k = \omega$ (critical damping - double real roots)

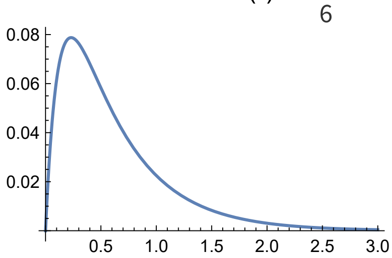
c) $k < \omega$ (underdamping - complex roots)

a) $k > \omega$ (overdamping - distinct real roots)

Example: $k = 5,\ \omega = 4$

$x''(t) + 10\,x'(t) + 16\,x(t) = 0,\quad x(0) = 0,\quad x'(0) = 1$

$\Longrightarrow x(t) = -\frac{1}{6}\,e^{-8\,t} + \frac{1}{6}\,e^{-2\,t}$

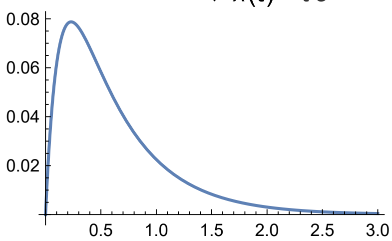


b) $k = \omega$ (critical damping - double real roots)

Example: $k = 2,\ \omega = 2$

$x''(t) + 4\,x'(t) + 4\,x(t) = 0,\quad x(0) = 0,\quad x'(0) = 1$

$\Longrightarrow x(t) = t\,e^{-2\,t}$

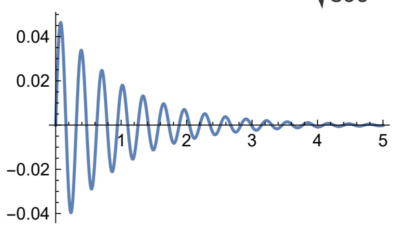


c) $k < \omega$ (underdamping - complex roots)

Example: $k = 1,\ \omega = 20$

$x''(t) + 2\,x'(t) + 400\,x(t) = 0,\quad x(0) = 0,\quad x'(0) = 1$

$\Longrightarrow x(t) = \frac{1}{\sqrt{399}}\,e^{-t}\sin\left(\sqrt{399}\,t\right)$



3. Forced harmonic motion

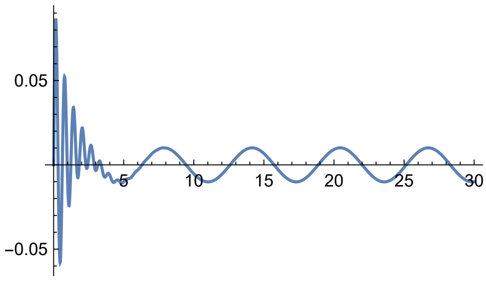
$$x''(t) + 2\,k\,x'(t) + \omega^2\,x(t) = \frac{F}{m}\sin(\alpha\,t)$$

$F\sin(\alpha\,t)$: external force or driving force,
α: driving frequency (frequency of the external force)

Example: $m = 1,\ k = 1,\ \omega = 10,\ F = 1,\ \alpha = 1$

$x''(t) + 2\,x'(t) + 100\,x(t) = \sin t,\quad x(0) = 0,\ x'(0) = 1$

$$\Longrightarrow\ x(t) = e^{-t}\left(\frac{2}{9805}\cos\left(3\sqrt{11}\,t\right) + \frac{3236}{9805\sqrt{11}}\sin\left(3\sqrt{11}\,t\right)\right) + \frac{-2\cos t + 99\sin t}{9805}$$



4. Forced undamped harmonic motion

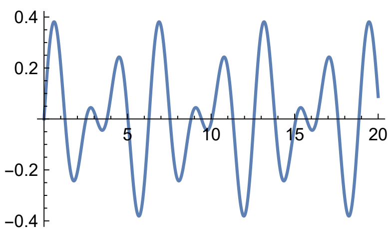
$$x''(t) + \omega^2\,x(t) = \frac{F}{m}\sin(\alpha\,t)$$

ω: natural frequency of the system (the frequency at which a system tends to oscillate in the absence of any driving or damping force)
α: driving frequency (frequency of the external force)

Example: $m = 1,\ \omega = 3,\ F = 1,\ \alpha = 2$

$x''(t) + 9\,x(t) = \sin 2\,t,\quad x(0) = 0,\ x'(0) = 1$

$$\Longrightarrow\ x(t) = \frac{1}{5}(\sin 2\,t + \sin 3\,t)$$

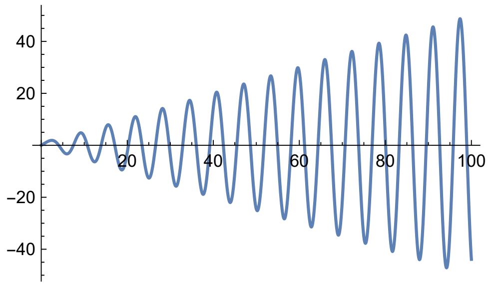


Resonance: $\omega = \alpha$

Example: $m = 1,\ \omega = 1,\ F = 1,\ \alpha = 1$

$$x''(t) + x(t) = \sin t, \quad x(0) = 0, \quad x'(0) = 1$$

$$\Longrightarrow x(t) = \frac{1}{2}(-t\cos t + 3\sin t)$$



Examples: - pushing a person in a swing
- electrical resonance of tuned circuits in radios and TVs that allow radio frequencies to selectively received
- acoustic resonances of musical instruments etc.

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Hyperlink["https://en.wikipedia.org/wiki/Resonance"]
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https://en.wikipedia.org/wiki/Resonance