04 - Higher order linear differential equations, solutions

1. Homogeneous equations, general solutions

a) $y'' - 8y' + 15y = 0$

Characteristic equation: $\lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5) = 0 \implies \lambda_1 = 3,\ \lambda_2 = 5$ (single real roots)
Linearly independent solutions: $e^{3t},\ e^{5t}$
The general solution: $y(t) = c_1 e^{3t} + c_2 e^{5t}$

b) $y'' + 2y' = 0$

Characteristic equation: $\lambda^2 + 2\lambda = \lambda(\lambda + 2) = 0 \implies \lambda_1 = 0,\ \lambda_2 = -2$ (single real roots)
Linearly independent solutions: $e^{0t} = 1,\ e^{-2t}$
The general solution: $y(t) = c_1 + c_2 e^{-2t}$

c) $y'' - 8y' + 16y = 0$

Characteristic equation: $\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$
$\implies \lambda_1 = \lambda_2 = 4$ (double real roots, inner resonance)
Linearly independent solutions: $e^{4t},\ t e^{4t}$
The general solution: $y(t) = c_1 e^{4t} + c_2 t e^{4t}$

d) $y'' + 4y' + 13y = 0$

Characteristic equation: $\lambda^2 + 4\lambda + 13 = 0$
$$\implies \lambda_{1,2} = \frac{-4 \pm \sqrt{16 - 4 \cdot 13}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i \text{ (complex roots)}$$
Linearly independent solutions: $e^{-2t} \cos 3t,\ e^{-2t} \sin 3t$
The general solution: $y(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t$

e) $y'' + 25y = 0$

Characteristic equation: $\lambda^2 + 25 = 0 \implies \lambda^2 = -25 \implies \lambda_{1,2} = \pm 5i = 0 \pm 5i$ (complex roots)
Linearly independent solutions: $e^{0t} \cos 5t = \cos 5t,\ e^{0t} \sin 5t = \sin 5t$
The general solution: $y(t) = c_1 \cos 5t + c_2 \sin 5t$

f) $y''' + 2y'' + y' = 0$

Characteristic equation: $\lambda^3 + 2\lambda^2 + \lambda = \lambda(\lambda + 1)^2 = 0$
$\implies \lambda_1 = 0,\ \lambda_2 = \lambda_3 = -1$ (one single real root and double real roots, inner resonance)
Linearly independent solutions: $e^{0t} = 1,\ e^{-t},\ t e^{-t}$
The general solution: $y(t) = c_1 + c_2 e^{-t} + c_3 t e^{-t}$

g) $y^{(4)} - y = 0$

Characteristic equation: $\lambda^4 - 1 = (\lambda^2 - 1)(\lambda^2 + 1) = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i) = 0$

$\implies \lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 0 + i,\ \lambda_4 = 0 - i$ (single real roots and single complex roots)

Linearly independent solutions: $e^t,\ e^{-t},\ e^{0t}\cos t = \cos t,\ e^{0t}\sin t = \sin t$

The general solution: $y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

h) $y^{(4)} - y''' = 0$

Characteristic equation: $\lambda^4 - \lambda^3 = \lambda^3(\lambda - 1) = 0$

$\implies \lambda_1 = 1,\ \lambda_2 = \lambda_3 = \lambda_4 = 0$ (one single real root and multiple real roots, inner resonance)

Linearly independent solutions: $e^t,\ e^{0t} = 1,\ t e^{0t} = t,\ t^2 e^{0t} = t^2$

The general solution: $y(t) = c_1 e^t + c_2 + c_3 t + c_4 t^2$

2. Homogeneous equations, initial value problems

a) $y'' + 2y' + 2y = 0,\ y(0) = 2,\ y'(0) = 1$

Characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \implies \lambda_{1,2} = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$ (complex roots)

Linearly independent solutions: $e^{-t}\cos t,\ e^{-t}\sin t$

The general solution: $y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

The derivative of the general solution: $y'(t) = -c_1 e^{-t}\cos t - c_1 e^{-t}\sin t - c_2 e^{-t}\sin t + c_2 e^{-t}\cos t$

From the initial conditions: $y(0) = 2 \implies c_1 \cdot 1 + c_2 \cdot 0 = 2$

$y'(0) = 1 \implies -c_1 \cdot 1 - c_1 \cdot 0 - c_2 \cdot 0 + c_2 \cdot 1 = 1$

$\implies c_1 = 2,\ c_2 = 3$

The solution of the initial value problem: $y(t) = 2e^{-t}\cos t + 3e^{-t}\sin t$

b) $y'' + 3y' - 4y = 0,\ y(0) = 3,\ y'(0) = -4$

Characteristic equation: $\lambda^2 + 3\lambda - 4 = (\lambda + 4)(\lambda - 1) = 0 \implies \lambda_1 = 1,\ \lambda_2 = -4$ (single real roots)

Linearly independent solutions: $e^t,\ e^{-4t}$

The general solution: $y(t) = c_1 e^t + c_2 e^{-4t}$

The derivative of the general solution: $y'(t) = c_1 e^t - 4c_2 e^{-4t}$

From the initial conditions: $y(0) = 3 \implies c_1 \cdot 1 + c_2 \cdot 1 = 3$

$y'(0) = -4 \implies c_1 \cdot 1 - 4c_2 \cdot 1 = -4$

$\implies c_1 = \dfrac{8}{5},\ c_2 = \dfrac{7}{5}$

The solution of the initial value problem: $y(t) = \dfrac{8}{5}e^t + \dfrac{7}{5}e^{-4t}$

c) $y'' + 10y' + 25y = 0,\ y(0) = -1,\ y'(0) = 7$

Characteristic equation: $\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0$
$\Longrightarrow\ \lambda_1 = \lambda_2 = -5$ (double real roots, inner resonance)
Linearly independent solutions: $e^{-5t},\ t e^{-5t}$
The general solution: $y(t) = c_1 e^{-5t} + c_2 t e^{-5t}$

The derivative of the general solution: $y'(t) = -5c_1 e^{-5t} + c_2 e^{-5t} - 5c_2 t e^{-5t}$

From the initial conditions: $y(0) = -1 \Longrightarrow c_1 \cdot 1 + c_2 \cdot 0 = -1$
$y'(0) = 7 \Longrightarrow -5c_1 \cdot 1 + c_2 \cdot 1 - 0 = 7$
$\Longrightarrow\ c_1 = -1,\ c_2 = 2$

The solution of the initial value problem: $y(t) = -e^{-5t} + 2t e^{-5t}$

3. Nonhomogeneous equations

a) $y'' - 5y' + 6y = 2\sin 2t$

The general solution of the homogeneous equation:
$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0 \Longrightarrow \lambda_1 = 2,\ \lambda_2 = 3, \Longrightarrow y_h = c_1 e^{2t} + c_2 e^{3t}$

Finding a particular solution of the nonhomogeneous equation:

$$\begin{aligned} 6\cdot &\,|\ y_p := A\sin 2t + B\cos 2t \\ -5\cdot &\,|\ y_p' = 2A\cos 2t - 2B\sin 2t \\ 1\cdot &\,|\ y_p'' = -4A\sin 2t - 4B\cos 2t \end{aligned}$$

Substituting into the nonhomogeneous equation:

$$(-4A + 10B + 6A)\sin 2t + (-4B - 10A + 6B)\cos 2t = 2\sin 2t$$

We want to find A and B such that the equation holds for all $t \in \mathbb{R}$
$\Longrightarrow$ since the sine and cosine functions are linearly independent
then by comparing the coefficients of the corresponding terms on both sides:

$$\begin{aligned} -4A + 10B + 6A = 2 &\Longrightarrow A + 5B = 1 \Longrightarrow A + 25A = 1 \Longrightarrow A = \frac{1}{26},\ B = \frac{5}{26} \\ -4B - 10A + 6B = 0 &\quad\ \ B - 5A = 0 \qquad B = 5A \end{aligned}$$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 e^{3t} + \frac{1}{26}\sin 2t + \frac{5}{26}\cos 2t \qquad (c_1,\ c_2 \in \mathbb{R})$$

b) $y'' - 5y' + 6y = 2\,t\,e^t$

The general solution of the homogeneous equation:
$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0 \implies \lambda_1 = 2,\ \lambda_2 = 3, \implies y_h = c_1 e^{2t} + c_2 e^{3t}$

Finding a particular solution of the nonhomogeneous equation:
$f(t) = 2\,t\,e^t$ is the product of a first-degree polynomial and an exponential function
$\implies y_p := (A\,t + B)\,e^t$

$$\begin{aligned} 6\cdot \mid\ & y_p := (A\,t + B)\,e^t = A\,t\,e^t + B\,e^t \\ -5\cdot \mid\ & y_p' = A\,e^t + A\,t\,e^t + B\,e^t \\ 1\cdot \mid\ & y_p'' = A\,e^t + A\,e^t + A\,t\,e^t + B\,e^t \end{aligned}$$

Substituting into the nonhomogeneous equation:

$$t\,e^t(A - 5A + 6A) + e^t(2A + B - 5A - 5B + 6B) = 2\,t\,e^t$$

Comparing the coefficients of the corresponding terms on both sides:

$$\begin{aligned} & A - 5A + 6A = 2 && \implies A = 1,\ B = \frac{3}{2} \\ & 2A + B - 5A - 5B + 6B = 0 && \implies -3A + 2B = 0 \end{aligned}$$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 e^{3t} + \left(t + \frac{3}{2}\right)e^t \qquad (c_1,\ c_2 \in \mathbb{R})$$

c) $y'' - 6y' + 13y = 39$

The general solution of the homogeneous equation:
$\lambda^2 - 6\lambda + 13 = 0 \implies \lambda_{12} = 3 \pm 2i \implies y_h = c_1 e^{3t}\cos 2t + c_2 e^{3t}\sin 2t$

Finding a particular solution of the nonhomogeneous equation:

$$\begin{aligned} 13\cdot \mid\ & y_p := A \\ -6\cdot \mid\ & y_p' = 0 \\ 1\cdot \mid\ & y_p'' = 0 \end{aligned}$$

Substituting into the nonhomogeneous equation:
$13A = 39 \implies A = 3$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{3t}\cos 2t + c_2 e^{3t}\sin 2t + 3 \qquad (c_1,\ c_2 \in \mathbb{R})$$

d) $y'' - y' - 2y = 3e^{2t}$

The general solution of the homogeneous equation:
$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \implies \lambda_1 = 2, \ \lambda_2 = -1, \implies y_h = c_1 e^{2t} + c_2 e^{-t}$

Finding y_p, first try, based on the form of the right-hand side $f(t) = 3e^{2t}$:

$$\begin{array}{r|l} -2\cdot & y_p := A e^{2t} \\ -1\cdot & y_p' = 2A e^{2t} \\ 1\cdot & y_p'' = 4A e^{2t} \end{array}$$

Substituting into the nonhomogeneous equation:

$e^{2t}(-2A - 2A + 4A) = 3e^{2t} \implies e^{2t} \cdot 0 = 3e^{2t}$, however, $3e^{2t} \neq 0$!

This is a contradiction which means that $y_p = A e^{2t}$ is not correct.
The problem is that the term $A e^{2t}$ in y_p is a constant multiple of the term $c_1 e^{2t}$ in y_h.
This is called an **outer resonance**. In this case we multiply the term $A e^{2t}$ in y_p by t until the resonance disappears.

The correct form for y_p:

$$\begin{array}{r|l} -2\cdot & y_p := \mathbf{A\,t\,e^{2t}} \Longleftarrow y_h = \mathbf{c_1 e^{2t}} + c_2 e^{-t} \\ -1\cdot & y_p' = A e^{2t} + 2A t e^{2t} \\ 1\cdot & y_p'' = 2A e^{2t} + 2A e^{2t} + 4A t e^{2t} \end{array}$$

Substituting into the nonhomogeneous equation:

$t e^{t}(4A - 2A - 2A) + e^{2t}(4A - A) = 3e^{2t}$

Comparing the coefficients of the corresponding terms on both sides:
$4A - A = 3 \implies A = 1$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 e^{-t} + t e^{2t} \qquad (c_1, \ c_2 \in \mathbb{R})$$

e) $y'' - 3y' + 2y = e^{3t} + 4t^2 - 6$

The general solution of the homogeneous equation:
$\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0 \implies \lambda_1 = 1, \ \lambda_2 = 2, \implies y_h = c_1 e^{t} + c_2 e^{2t}$

Finding a particular solution of the nonhomogeneous equation:
$f(t) = \left(e^{3t}\right) + \left(4t^2 - 6\right)$ is the sum of an exponential function and a quadratic polynomial $\implies$

$2\cdot\ |\ \ y_p := A\,e^{3t} + B\,t^2 + C\,t + D$

$-3\cdot\ |\ \ y_p{}' = 3\,A\,e^{3t} + 2\,B\,t + C$

$1\cdot\ |\ \ y_p{}'' = 9\,A\,e^{3t} + 2\,B$

Substituting into the nonhomogeneous equation:

$$e^{3t}(9\,A - 9\,A + 2\,A) + t^2(2\,B) + t(-6\,B + 2\,C) + (2\,B - 3\,C + 2\,D) = e^{3t} + 4\,t^2 - 6$$

Comparing the coefficients of the corresponding terms on both sides:

$9\,A - 9\,A + 2\,A = 1 \quad \Longrightarrow \quad A = \frac{1}{2}$

$2\,B = 4 \quad \Longrightarrow \quad B = 2$

$-6\,B + 2\,C = 0 \quad \Longrightarrow \quad C = 6$

$2\,B - 3\,C + 2\,D = -6 \quad \Longrightarrow \quad D = 4$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1\,e^t + c_2\,e^{2t} + \frac{1}{2}\,e^{3t} + 2\,t^2 + 6\,t + 4 \qquad (c_1,\ c_2 \in \mathbb{R})$$

f) $y'' - 3\,y' + 2\,y = t + e^t$

The general solution of the homogeneous equation:

$\lambda^2 - 3\,\lambda + 2 = (\lambda - 2)\,(\lambda - 1) = 0 \;\Longrightarrow\; \lambda_1 = 1,\ \lambda_2 = 2, \;\Longrightarrow\; y_h = \boldsymbol{c_1\,e^t} + c_2\,e^{2t}$

Finding a particular solution of the nonhomogeneous equation:

$2\cdot\ |\ \ y_p := (A\,t + B) + \boldsymbol{C\,t\,e^t}$ $\Longleftarrow$ **outer resonance**

$-3\cdot\ |\ \ y_p{}' = A + C\,e^t + C\,t\,e^t$

$1\cdot\ |\ \ y_p{}'' = C\,e^t + C\,e^t + C\,t\,e^t$

Substituting into the nonhomogeneous equation:

$$t\,e^t(2\,C - 3\,C + C) + e^t(-3\,C + 2\,C) + t(2\,A) + (2\,B - 3\,A) = t + e^t$$

Comparing the coefficients of the corresponding terms on both sides:

$-3\,C + 2\,C = 1 \quad \Longrightarrow \quad C = -1$

$2\,A = 1 \quad \Longrightarrow \quad A = \frac{1}{2}$

$2\,B - 3\,A = 0 \quad \Longrightarrow \quad B = \frac{3}{4}$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1\,e^t + c_2\,e^{2t} + \frac{1}{2}\,t + \frac{3}{4} - t\,e^t \qquad (c_1,\ c_2 \in \mathbb{R})$$

g) $y'' - 2y' + y = 6e^t$

The general solution of the homogeneous equation:

$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \implies \lambda_1 = \lambda_2 = 1, \implies y_h = \mathbf{C_1 e^t} + \mathbf{C_2\, t\, e^t}$ **(inner resonance)**

Finding a particular solution of the nonhomogeneous equation:

$y_p = \mathbf{A e^t}$ (not suitable because of $\mathbf{C_1 e^t}$ in y_h)

$y_p = \mathbf{A\, t\, e^t}$ (not suitable because of $\mathbf{C_2\, t\, e^t}$ in y_h)

The correct choice for y_p:

$$\begin{aligned} 1\cdot \,|\,\; & y_p := C\,t^2 e^t \quad \Longleftarrow \textbf{ outer resonance} \\ -2\cdot \,|\,\; & y_p' = 2\,C\,t\,e^t + C\,t^2 e^t \\ 1\cdot \,|\,\; & y_p'' = 2\,C\,e^t + 2\,C\,t\,e^t + 2\,C\,t\,e^t + C\,t^2 e^t \end{aligned}$$

Substituting into the nonhomogeneous equation:

$$t^2 e^t (C - 2C + C) + t\,e^t (4C - 4C) + e^t (2C) = 6e^t$$

Comparing the coefficients of the corresponding terms on both sides:

$2C = 6 \quad \implies \quad C = 3$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^t + c_2\, t\, e^t + 3\,t^2 e^t \qquad (c_1,\; c_2 \in \mathbb{R})$$

h) $y'' + 8y' + 25y = e^{-4t}$

The general solution of the homogeneous equation:

$$\lambda^2 + 8\lambda + 25 = 0 \implies \lambda_{1,2} = \frac{-8 \pm \sqrt{64 - 100}}{2} = -4 \pm 3i \implies y_h = c_1 e^{-4t} \cos 3t + c_2 e^{-4t} \sin 3t$$

The particular solution of the nonhomogeneous equation: $y_p := A e^{-4t}$

There is no outer resonance here since $A e^{-4t}$ is not a constant multiple of the terms in y_h.

$$\begin{aligned} 25\cdot \,|\,\; & y_p := A e^{-4t} \\ 8\cdot \,|\,\; & y_p' = -4A e^{-4t} \\ 1\cdot \,|\,\; & y_p'' = 16A e^{-4t} \end{aligned}$$

Substituting into the nonhomogeneous equation:

$$e^{-4t}(16A - 32A + 25A) = e^{-4t}$$

Comparing the coefficients of the corresponding terms on both sides:

$9A = 1 \quad \Longrightarrow \; A = \frac{1}{9}$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-4t} \cos 3t + c_2 e^{-4t} \sin 3t + \frac{1}{9} e^{-4t} \qquad (c_1,\; c_2 \in \mathbb{R})$$

i) $y'' + 2y' = 2t + 3$

The general solution of the homogeneous equation:
$\lambda^2 + 2\lambda = \lambda(\lambda + 2) = 0 \;\Longrightarrow\; \lambda_1 = 0,\;\; \lambda_2 = -2 \;\Longrightarrow\; y_h = \mathbf{c_1} + c_2 e^{-2t}$

Finding a particular solution of the nonhomogeneous equation:
$y_p = A t + \mathbf{B}$ (not suitable because of $\mathbf{c_1}$ in y_h)

The correct choice for y_p:

$$\begin{array}{r|l} 0\cdot & y_p := (At + B)\,t = A t^2 + B t \\ 2\cdot & y_p' = 2At + B \\ 1\cdot & y_p'' = 2A \end{array}$$

$\Longleftarrow$ **outer resonance**

Substituting into the nonhomogeneous equation:

$$t(4A) + (2A + 2B) = 2t + 3$$

Comparing the coefficients of the corresponding terms on both sides:
$4A = 2 \quad \Longrightarrow \; A = \frac{1}{2}$
$2A + 2B = 3 \;\Longrightarrow\; B = 1$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 + c_2 e^{-2t} + \frac{1}{2} t^2 + t \qquad (c_1,\; c_2 \in \mathbb{R})$$

j) $y'' + y = \sin t$

The general solution of the homogeneous equation:
$\lambda^2 + 1 = 0 \;\Longrightarrow\; \lambda_{1,2} = \pm i = 0 \pm i \;\Longrightarrow\; y_h = \mathbf{c_1 \cos t} + \mathbf{c_2 \sin t}$

Finding a particular solution of the nonhomogeneous equation:
$y_p = \mathbf{A \cos t} + \mathbf{B \sin t}$ (not suitable because of the terms in y_h)

The correct choice for y_p:

$$1\cdot\;|\quad y_p := \mathbf{A\,t\cos t + B\,t\sin t} \quad \Longleftarrow \textbf{ outer resonance}$$
$$0\cdot\;|\quad y_p{}' = A\cos t - A\,t\sin t + B\sin t + B\,t\cos t$$
$$1\cdot\;|\quad y_p{}'' = \mathbf{-A\sin t - A\sin t - A\,t\cos t + B\cos t + B\cos t - B\,t\sin t}$$

Substituting into the nonhomogeneous equation:

$$\mathbf{t\cos t(A - A) + t\sin t(B - B) + \cos t(2\,B) + \sin t(-2\,A)} = \sin t$$

Comparing the coefficients of the corresponding terms on both sides:

$2\,B = 0 \quad \Longrightarrow \quad B = 0$

$-2\,A = 1 \quad \Longrightarrow \quad A = -\frac{1}{2}$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1\cos t + c_2\sin t - \frac{1}{2}\,t\cos t \qquad (c_1,\ c_2 \in \mathbb{R})$$

4. Nonhomogeneous equations

$$L\,I''(t) + R\,I'(t) + \frac{1}{C}\,I(t) = F(t) \text{ where } L = 1,\ R = 3,\ C = \frac{1}{2} \implies I''(t) + 3\,I'(t) + 2\,I(t) = F(t)$$

$$\lambda^2 + 3\,\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0 \implies \lambda_1 = -1,\ \lambda_1 = -2$$

The general solution of the homogeneous equation:

$\mathbf{I_h(t) = c_1\,e^{-t} + c_2\,e^{-2\,t}}$

a) $F(t) = e^t \implies I_p(t) = A\,e^t$

Result: $I(t) = I_h(t) + I_p(t) = c_1\,e^{-t} + c_2\,e^{-2\,t} + \frac{1}{6}\,e^t$

b) $F(t) = e^{-t} \implies I_p(t) = A\,t\,e^{-t}$ (outer resonance)

Result: $I(t) = I_h(t) + I_p(t) = c_1\,e^{-t} + c_2\,e^{-2\,t} + t\,e^{-t}$

c) $F(t) = 2\,t + 1 \implies I_p(t) = A\,t + B$

Result: $I(t) = I_h(t) + I_p(t) = c_1\,e^{-t} + c_2\,e^{-2\,t} + t - 1$

d) $F(t) = t \implies I_p(t) = A\,t + B$

Result: $I(t) = I_h(t) + I_p(t) = c_1\,e^{-t} + c_2\,e^{-2\,t} + \frac{t}{2} - \frac{3}{4}$

e) $F(t) = t^2 \implies I_p(t) = A\,t^2 + B\,t + C$

Result: $I(t) = I_h(t) + I_p(t) = c_1\, e^{-t} + c_2\, e^{-2\,t} + \frac{t^2}{2} - \frac{3\,t}{2} + \frac{7}{4}$

f) $F(t) = 3 \implies I_p(t) = A$

Result: $I(t) = I_h(t) + I_p(t) = c_1\, e^{-t} + c_2\, e^{-2\,t} + \frac{3}{2}$

g) $F(t) = \sin 2\,t \implies I_p(t) = A \sin 2\,t + B \cos 2\,t$

Result: $I(t) = I_h(t) + I_p(t) = c_1\, e^{-t} + c_2\, e^{-2\,t} - \frac{1}{20} \sin 2\,t - \frac{3}{20} \cos 2\,t$

h) $F(t) = e^{-t} \sin 2\,t \implies I_p(t) = e^{-t}(A \sin 2\,t + B \cos 2\,t)$ (no outer resonance)

Result: $I(t) = I_h(t) + I_p(t) = c_1\, e^{-t} + c_2\, e^{-2\,t} - \frac{1}{5}\, e^{-t} \sin 2\,t - \frac{1}{10}\, e^{-t} \cos 2\,t$

i) $F(t) = e^{2\,t} - \cos t \implies I_p(t) = A\, e^{2\,t} + B \sin t + C \cos t$

Result: $I(t) = I_h(t) + I_p(t) = c_1\, e^{-t} + c_2\, e^{-2\,t} + \frac{1}{12}\, e^{2\,t} - \frac{3}{10} \sin t - \frac{1}{10} \cos t$

j) $F(t) = 4 + e^{-t} \implies I_p(t) = A + B\, t\, e^{-t}$ (outer resonance)

Result: $I(t) = I_h(t) + I_p(t) = c_1\, e^{-t} + c_2\, e^{-2\,t} + t\, e^{-t} + 2$