03 - First order linear differential equations

Solution method

Definition. A first order linear differential equation has the form

$$y'(t) + f(t)\,y(t) = g(t)$$

where f and g are continuous functions on an interval I.
If $g(t) \equiv 0$ then the equation is said to be homogeneous, otherwise it is said to be nonhomogeneous.

Solution. 1st step: Consider the homogeneous equation

$$y'(t) + f(t)\,y(t) = 0 \implies \frac{dy}{dt} = -f(t)\,y$$

It is a separable equation. The constant solution is $y(t) \equiv 0$. If $y \neq 0$ then by separating the variables, we get $\int \frac{1}{y}\,dy = -\int f(t)\,dt$. Let $F(t) = \int f(t)\,dt$, then

$\ln|y| = -F(t) + c_1 \implies |y| = e^{c_1} e^{-F(t)} = K \cdot e^{-F(t)}$, where $K > 0$.
If $y > 0$ then $y = K \cdot e^{-F(t)}$, if $y < 0$ then $y = -K \cdot e^{-F(t)}$ and $y \equiv 0$ is also a solution.

Therefore, **the general solution of the homogeneous equation** is

$y_h(t) = C \cdot e^{-F(t)}$ where $C \in \mathbb{R}$.

2nd step: variation of the constant method. We suppose that the nonhomogeneous equation has a **particular solution** of the form

$$y_p(t) = c(t) \cdot e^{-F(t)} \implies y_p'(t) = c'(t) \cdot e^{-F(t)} + c(t) \cdot e^{-F(t)} \cdot (-f(t))$$

Substituting $y_p'(t)$ into the nonhomogeneous equation $y'(t) + f(t)\,y(t) = g(t)$, we get

$$c'(t) \cdot e^{-F(t)} + c(t) \cdot e^{-F(t)} \cdot (-f(t)) + f(t) \cdot c(t) \cdot e^{-F(t)} = g(t)$$

$$\implies c'(t) \cdot e^{-F(t)} = g(t)$$

$$\implies c'(t) = g(t) \cdot e^{F(t)} \implies c(t) = \int g(t) \cdot e^{F(t)}\,dt + D$$

The general solution of the nonhomogeneous equation is

$$y(t) = y_p(t) + y_h(t) = \left(\int g(t) \cdot e^{F(t)}\,dt\right) e^{-F(t)} + D\,e^{-F(t)} \quad \text{(we obtain } y_p(t) \text{ when } D = 0\text{)}$$