03 - First order linear differential equations, solutions

1. In the mixing problem, suppose that the salt content in the inflow decreases exponentially. Then the equation is $y'(t) = 0.6\, e^{-t} - 0.2\, y(t)$

Solution. 1st step: The homogeneous equation is

$$y'(t) = -0.2\, y(t) \Longrightarrow \frac{dy}{dt} = -0.2\, y \quad \text{(separable)}$$

Constant solution: $y \equiv 0$. If $y \neq 0$ then

$$\int \frac{1}{y}\, dy = \int -0.2\, dt \Longrightarrow \ln|y| = -0.2\, t + c_1$$

$$\Longrightarrow |y| = e^{-0.2\, t + c_1} \Longrightarrow y = \pm e^{c_1} \cdot e^{-0.2\, t} \text{ or } y \equiv 0.$$

The general solution of the homogeneous equation is

$$y_h(t) = C \cdot e^{-0.2\, t}, \quad \text{where } C \in \mathbb{R}.$$

2nd step, variation of the constant method: The particular solution of the nonhomogeneous equation is

$$y_p(t) = c(t) \cdot e^{-0.2\, t} \Longrightarrow y_p'(t) = c'(t) \cdot e^{-0.2\, t} + c(t) \cdot e^{-0.2\, t} \cdot (-0.2)$$

Substituting into the nonhomogeneous equation $y'(t) = 0.6\, e^{-t} - 0.2\, y(t)$, we get

$$c'(t) \cdot e^{-0.2\, t} + c(t) \cdot e^{-0.2\, t} \cdot (-0.2) = 0.6 \cdot e^{-t} - 0.2\, c(t) \cdot e^{-0.2\, t}$$

$$\Longrightarrow c'(t) \cdot e^{-0.2\, t} = 0.6 \cdot e^{-t}$$

$$\Longrightarrow c'(t) = 0.6\, e^{-0.8\, t} \Longrightarrow c(t) = \int 0.6\, e^{-0.8\, t}\, dt = 0.6 \cdot \frac{e^{-0.8\, t}}{-0.8} = -0.75\, e^{-0.8\, t}$$

$$\Longrightarrow y_p(t) = c(t) \cdot e^{-0.2\, t} = -0.75\, e^{-0.8\, t} \cdot e^{-0.2\, t} = -0.75\, e^{-t}$$

The general solution of the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = C \cdot e^{-0.2\, t} - 0.75\, e^{-t}, \text{ where } C \in \mathbb{R}$$

2. $y'(t) = \dfrac{y(t)}{t} + t \qquad (t \neq 0)$

Solution. 1st step: The homogeneous equation is

$$y'(t) = \frac{y(t)}{t} \Longrightarrow \frac{dy}{dt} = \frac{y}{t} \quad \text{(separable)}$$

Constant solution: $y \equiv 0$. If $y \neq 0$ then

$\int \frac{1}{y}\,\mathrm{dy} = \int \frac{1}{t}\,\mathrm{dt} \Longrightarrow \ln|y| = \ln|t| + c_1$

$\Longrightarrow |y| = e^{\ln|t|+c_1} = e^{c_1} e^{\ln|t|} = e^{c_1}|t|$

$\Longrightarrow y = \pm e^{c_1} \cdot t$ or $y \equiv 0.$

The general solution of the homogeneous equation is

$y_h(t) = C \cdot t$, where $C \in \mathbb{R}$.

2nd step, variation of the constant method: The particular solution of the nonhomogeneous equation is

$y_p(t) = c(t) \cdot t \Longrightarrow y_p'(t) = c'(t) \cdot t + c(t) \cdot 1$

Substituting into the nonhomogeneous equation $y'(t) = \frac{y(t)}{t} + t$, we get

$c'(t) \cdot t + c(t) = \frac{c(t) \cdot t}{t} + t$

$\Longrightarrow c'(t) \cdot t = t$

$\Longrightarrow c'(t) = 1 \Longrightarrow c(t) = \int 1\,\mathrm{dt} = t$

$\Longrightarrow y_p(t) = c(t) \cdot t = t \cdot t = t^2$

The general solution of the nonhomogeneous equation is

$y(t) = y_h(t) + y_p(t) = C \cdot t + t^2$, where $C \in \mathbb{R}$

3. $x'(t) + 2x(t) = e^t$, $x(0) = 0$

Solution. 1st step: The homogeneous equation is

$x'(t) + 2x(t) = 0 \Longrightarrow \frac{\mathrm{dx}}{\mathrm{dt}} = -2x$ (separable)

Constant solution: $x \equiv 0$. If $x \neq 0$ then

$\int \frac{1}{x}\,\mathrm{dx} = \int -2\,\mathrm{dt} \Longrightarrow \ln|x| = -2t + c_1$

$\Longrightarrow |x| = e^{-2t+c_1} = e^{c_1} e^{-2t}$

$\Longrightarrow x = \pm e^{c_1} \cdot e^{-2t}$ or $x \equiv 0.$

The general solution of the homogeneous equation is

$x_h(t) = C \cdot e^{-2t}$, where $C \in \mathbb{R}$.

2nd step, variation of the constant method: The particular solution of the nonhomogeneous equation is

$x_p(t) = c(t)\cdot e^{-2t} \implies x_p'(t) = c'(t)\cdot e^{-2t} + c(t)\cdot e^{-2t}(-2)$

Substituting into the nonhomogeneous equation $x'(t) + 2x(t) = e^t$, we get

$c'(t)\cdot e^{-2t} + c(t)\cdot e^{-2t}(-2) + 2c(t)\cdot e^{-2t} = e^t$

$\implies c'(t)\cdot e^{-2t} = e^t$

$\implies c'(t) = e^{3t} \implies c(t) = \int e^{3t}\,\mathrm{dt} = \frac{e^{3t}}{3}$

$\implies x_p(t) = c(t)\cdot e^{-2t} = \frac{e^{3t}}{3}\cdot e^{-2t} = \frac{1}{3}e^t$

The general solution of the nonhomogeneous equation is

$x(t) = x_h(t) + x_p(t) = C\cdot e^{-2t} + \frac{1}{3}e^t$, where $C \in \mathbb{R}$

From the initial condition $x(0) = 0$

$\implies 0 = C\cdot e^0 + \frac{1}{3}\cdot e^0 \implies C = -\frac{1}{3}$

The solution of the initial value problem is $x(t) = -\frac{1}{3}\cdot e^{-2t} + \frac{1}{3}e^t$

4. $t\,x'(t) - 2x(t) = 2t^4$

Solution. Dividing by t, the equation is $x'(t) - \frac{2}{t}x(t) = 2t^3$

1st step: $x'(t) - \frac{2}{t}x(t) = 0 \implies \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{2}{t}x$ (separable)

Constant solution: $x \equiv 0$. If $x \not\equiv 0$ then

$\int \frac{1}{x}\,\mathrm{dx} = \int \frac{2}{t}\,\mathrm{dt} \implies \ln|x| = 2\ln|t| + c_1$

$\implies |x| = e^{2\ln|t| + c_1} = e^{c_1}e^{2\ln|t|} = e^{c_1}e^{\ln|t|^2} = e^{c_1}|t|^2 = e^{c_1}\cdot t^2$

$\implies x = \pm e^{c_1}\cdot t^2$ or $x \equiv 0$.

The general solution of the homogeneous equation is

$x_h(t) = C\cdot t^2$, where $C \in \mathbb{R}$.

2nd step: The particular solution of the nonhomogeneous equation is

$x_p(t) = c(t)\cdot t^2 \implies x_p'(t) = c'(t)\cdot t^2 + c(t)\cdot 2t$

Substituting into the nonhomogeneous equation $x'(t) - \frac{2}{t}x(t) = 2t^3$, we get

$c'(t)\cdot t^2 + c(t)\cdot 2t - \frac{2}{t}c(t)\cdot t^2 = 2t^3$

$\implies c'(t)\cdot t^2 = 2t^3$

$$\Longrightarrow c'(t) = 2t \Longrightarrow c(t) = \int 2t\,\mathrm{dt} = t^2$$

$$\Longrightarrow x_p(t) = c(t) \cdot t^2 = t^2 \cdot t^2 = t^4$$

The general solution of the nonhomogeneous equation is

$x(t) = x_h(t) + x_p(t) = C \cdot t^2 + t^4$, where $C \in \mathbb{R}$

5. $E'(r) = -\frac{2}{r} E(r) + \frac{1}{r}$

Solution. 1st step: The homogeneous equation is

$E'(r) = -\frac{2}{r} E(r) \Longrightarrow \frac{\mathrm{dE}}{\mathrm{dr}} = -\frac{2}{r} E$ (separable)

Constant solution: $E \equiv 0$. If $E \not\equiv 0$ then

$$\int \frac{1}{E}\,\mathrm{dy} = \int -\frac{2}{r}\,\mathrm{dr} \Longrightarrow \ln|E| = -2\ln|r| + c_1$$

$$\Longrightarrow |E| = e^{-2\ln|r| + c_1} = e^{c_1} e^{-2\ln|r|} = e^{c_1} e^{\ln|r|^{-2}} = e^{c_1} |r|^{-2} = e^{c_1} \cdot \frac{1}{r^2}$$

$$\Longrightarrow E = \pm e^{c_1} \cdot \frac{1}{r^2} \quad \text{or} \quad E \equiv 0.$$

The general solution of the homogeneous equation is

$E_h(t) = C \cdot \frac{1}{r^2}$, where $C \in \mathbb{R}$.

2nd step: The particular solution of the nonhomogeneous equation is

$$E_p(t) = c(r) \cdot \frac{1}{r^2} \Longrightarrow E_p'(t) = c'(r) \cdot \frac{1}{r^2} + c(r) \cdot (-2)\, r^{-3}$$

Substituting into the nonhomogeneous equation $E'(r) = -\frac{2}{r} E(r) + \frac{1}{r}$, we get

$$c'(r) \cdot \frac{1}{r^2} + c(r) \cdot (-2)\, r^{-3} = -\frac{2}{r} \cdot c(r) \cdot \frac{1}{r^2} + \frac{1}{r}$$

$$\Longrightarrow c'(r) \cdot \frac{1}{r^2} = \frac{1}{r}$$

$$\Longrightarrow c'(r) = r \Longrightarrow c(t) = \int r\,\mathrm{dr} = \frac{r^2}{2}$$

$$\Longrightarrow E_p(t) = c(r) \cdot \frac{1}{r^2} = \frac{r^2}{2} \cdot \frac{1}{r^2} = \frac{1}{2}$$

The general solution of the nonhomogeneous equation is

$E(r) = E_h(t) + E_p(t) = C \cdot \frac{1}{r^2} + \frac{1}{2}$, where $C \in \mathbb{R}$

6.* The current $I(t)$ in an RC circuit is described by the equation $R\,I'(t) + \frac{1}{C}\,I(t) = F(t)$ where R, $C > 0$ are constants (R is the resistance and C is the capacity) and $F(t)$ is the external excitation.
a) Find the general solution if $I(0) = I_0$ and there is no external excitation, that is, $F(t) \equiv 0$.
b) Find the general solution if $R = C = 1$ and $F(t) = F_0 \sin t$ is a periodic excitation, where $F_0 > 0$ is a constant. Show that after a long time $I(t)$ can also be considered periodic.

Solution.

a) $R\,I'(t) + \frac{1}{C}\,I(t) = 0 \implies \frac{dI}{dt} = -\frac{1}{RC}\,I$

Constant solution: $I \equiv 0$. If $I \neq 0$ then separating the variables:

$$\int \frac{1}{I}\,dI = \int -\frac{1}{RC}\,dt \implies \ln|I| = -\frac{1}{RC}\,t + d_1 \implies |I| = e^{-\frac{1}{RC}t + d_1} = e^{d_1} \cdot e^{-\frac{1}{RC}t}$$

$\implies I = \pm e^{d_1} \cdot e^{-\frac{1}{RC}t}$ or $I \equiv 0$.

The general solution of the homogeneous equation: $I_h(t) = D \cdot e^{-\frac{1}{RC}t}$, $D \in \mathbb{R}$.
From the initial condition: $I(0) = I_0 \implies I_0 = D \cdot e^0 \implies D = I_0$
The solution of the initial value problem: $I_h(t) = I_0 \cdot e^{-\frac{1}{RC}t}$.

b) The equation is $I'(t) + I(t) = F_0 \sin t$.
Homogeneous equation: $I'(t) + I(t) = 0$
The general solution of the homogeneous equation: $I_h(t) = C \cdot e^{-t}$
The particular solution of the nonhomogeneous equation:
$I_p(t) = c(t) \cdot e^{-t} \implies I_p'(t) = c'(t) \cdot e^{-t} + c(t) \cdot e^{-t}(-1)$

$\implies c'(t) \cdot e^{-t} + c(t) \cdot e^{-t}(-1) + c(t) \cdot e^{-t} = F_0 \sin t$
$\implies c'(t)\,e^{-t} = F_0 \sin t$
$\implies c'(t) = F_0 \sin t \cdot e^t \implies c(t) = \int F_0 \sin t \cdot e^t\,dt$

The integral $I = \int e^t \cdot \sin t\,dt$ can be calculated by applying the integration by parts method twice, first with the choice $f'(t) = e^t$, $g(t) = \sin t$ and then with the choice $f'(t) = \sin t$, $g(t) = e^t$:

(1) $I = \int e^t \cdot \sin t\,dt = e^t \cdot \sin t - \int e^t \cdot \cos t\,dt$

(2) $I = \int e^t \cdot \sin t\,dt = e^t \cdot (-\cos t) - \int e^t \cdot (-\cos t)\,dt$

Adding the two equations and dividing by 2 we get: $I = \frac{e^t}{2}(\sin t - \cos t) + D$, $D \in \mathbb{R}$.

$$\Longrightarrow\ c(t) = \int F_0 \sin t \cdot e^t\, dt = \frac{F_0 \cdot e^t}{2}(\sin t - \cos t)\ \Longrightarrow\ I_p(t) = c(t)\, e^{-t} = \frac{F_0}{2}(\sin t - \cos t)$$

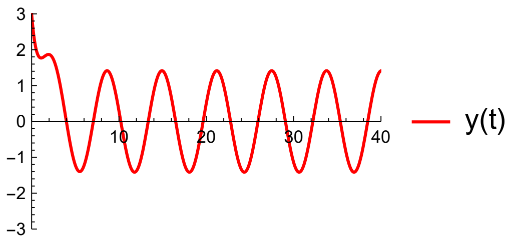
The general solution of the nonhomogeneous equation:

$I(t) = I_h(t) + I_p(t) = C \cdot e^{-t} + \frac{F_0}{2}(\sin t - \cos t)$

If $t \longrightarrow \infty$ then $\lim\limits_{t\to\infty} C \cdot e^{-t} = 0$, so after a long time the solution is approximately

$I(t) \approx \frac{F_0}{2}(\sin t - \cos t)$, which is periodic.

For example, the solution with $I(0) = 3$ and $F_0 = 2$ is $I(t) = 4\, e^{-t} - \cos t + \sin t$



Remark. In this example finding the particular solution will be simpler with the method of undetermined coefficients, see the lecture notes for topic 04.

7.* In the chemical reaction $X \xrightarrow{k} Y \xrightarrow{m} Z$, let $x(t)$, $y(t)$ and $z(t)$ denote the concentrations of the species X, Y and Z as a function of t, respectively. The reaction is described by the following differential equation system:

$x'(t) = -k\, x(t)$
$y'(t) = k\, x(t) - m\, y(t)$
$z'(t) = m\, y(t)$

where $k > 0$ and $m > 0$ are the reaction rate coefficients.

a) Solve the equation system if $k > m$ and $x(0) = 1$, $y(0) = 0$, $z(t) = 0$.
b) Show that $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} y(t) = 0$ and $\lim\limits_{t\to\infty} z(t) = 1$.

Solution. a)

(1)	$x'(t) = -k\, x(t)$	$x(0) = 1$
(2)	$y'(t) = k\, x(t) - m\, y(t)$	$y(0) = 0$
(3)	$z'(t) = m\, y(t)$	$z(0) = 0$

Equation (1) is separable: $x'(t) = -k\, x(t)$

$\Longrightarrow$ the general solution is $x(t) = c\, e^{-k t}$

If $x(0) = 1$ then $c = 1$.

$\Longrightarrow$ the solution of the initial value problem is $x(t) = e^{-k t}$

Substituting into (2), we get a first order linear nonhomogeneous differential equation.

$$y'(t) + m\,y(t) = k\,e^{-k\,t}$$

1st step: the homogeneous equation: $y'(t) + m\,y(t) = 0 \implies y'(t) = -m\,y(t)$
the general solution of the homogeneous equation: $y_h(t) = C\,e^{-m\,t}$

2nd step: the particular solution of the nonhomogeneous equation:
$y_p(t) = c(t)\,e^{-m\,t} \implies y_p'(t) = c'(t)\,e^{-m\,t} + c(t)\,e^{-m\,t}(-m)$

Substituting into the original equation $y'(t) + m\,y(t) = k\,e^{-k\,t}$, we get

$$c'(t)\,e^{-m\,t} + c(t)\,e^{-m\,t}(-m) + m\,c(t)\,e^{-m\,t} = k\,e^{-k\,t}$$
$$c'(t)\,e^{-m\,t} = k\,e^{-k\,t}$$
$$c'(t) = k\,e^{(-k+m)\,t}$$

If $k = m$: $c'(t) = k \implies c(t) = k\,t \implies y_p(t) = k\,t\,e^{-k\,t}$

The general solution of the nonhomogeneous equation:

$$y(t) = y_h(t) + y_p(t) = C\,e^{-k\,t} + k\,t\,e^{-k\,t}$$
$$y(0) = 0 \implies C = 0$$

The solution of the initial value problem:

$$y(t) = k\,t\,e^{-k\,t}$$

If $k \neq m$: $c'(t) = k\,e^{(-k+m)\,t} \implies c(t) = k \cdot \frac{e^{(-k+m)\,t}}{-k+m}$

$$\implies y_p(t) = k \cdot \frac{e^{(-k+m)\,t}}{-k+m} \cdot e^{-m\,t} = k \cdot \frac{e^{-k\,t}}{-k+m}$$

The general solution of the nonhomogeneous equation:

$$y(t) = y_h(t) + y_p(t) = C\,e^{-m\,t} + k \cdot \frac{e^{-k\,t}}{-k+m}$$
$$y(0) = 0 \implies C = -\frac{k}{-k+m} = \frac{k}{k-m}$$

The solution of the initial value problem:

$$y(t) = \frac{k}{k-m}\left(e^{-m\,t} - e^{-k\,t}\right)$$

Substituting into (3), we get a directly integrable equation:

$$z'(t) = m\,y(t) = \frac{k\,m}{k-m}\left(e^{-m\,t} - e^{-k\,t}\right) \implies z(t) = \frac{k\,m}{k-m}\left(\frac{e^{-m\,t}}{-m} - \frac{e^{-k\,t}}{-k}\right) + c$$

$$z(0) = 0 \implies c = -\frac{k\,m}{k-m}\left(\frac{1}{-m} - \frac{1}{-k}\right) = -\frac{k\,m}{k-m}\left(\frac{1}{k} - \frac{1}{m}\right) = -\frac{k\,m}{k-m} \cdot \frac{m-k}{k\,m} = 1$$

The solution of the initial value problem:

$$z(t) = \frac{k\,m}{k-m}\left(\frac{e^{-m\,t}}{-m} - \frac{e^{-k\,t}}{-k}\right) + 1 = -\frac{k}{k-m}\,e^{-m\,t} + \frac{m}{k-m}\,e^{-k\,t} + 1$$

Remark: (1) $x'(t) = -k\,x(t)$ $\quad x(0) = 1$

(2) $y'(t) = k\,x(t) - m\,y(t)$ $\quad y(0) = 0$

(3) $z'(t) = m\,y(t)$ $\quad z(0) = 0$

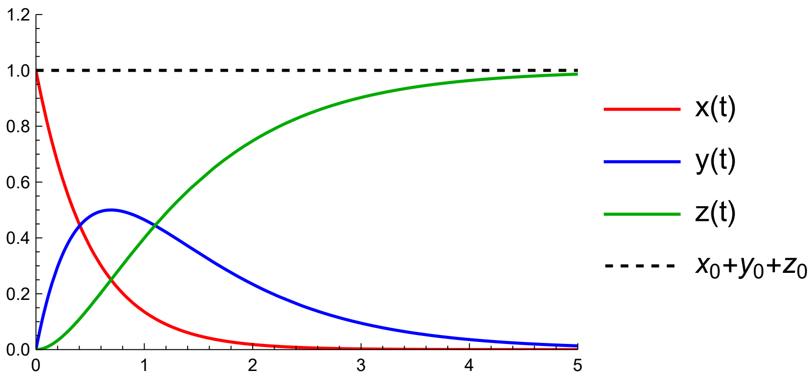
$\Longrightarrow x'(t) + y'(t) + z'(t) = 0$ for all $t \in \mathbb{R}$

$\Longrightarrow x(t) + y(t) + z(t) =$ constant for all $t \in \mathbb{R}$

Since $x(0) + y(0) + z(0) = 1$ then $x(t) + y(t) + z(t) = 1$ for all $t \in \mathbb{R}$

$$\Longrightarrow z(t) = 1 - x(t) - y(t) = 1 - e^{-k\,t} - \frac{k}{k-m}\left(e^{-m\,t} - e^{-k\,t}\right)$$

Plotting the solutions if $x_0 = 1,\ y_0 = 0,\ z_0 = 0,\ k = 2,\ m = 1$:



b) Show that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} z(t) = 1$

$x(t) = e^{-k\,t} \Longrightarrow \lim_{t\to\infty} x(t) = 0$

If $k = m$ then $y(t) = k\,t\,e^{-k\,t} \Longrightarrow \lim_{t\to\infty} y(t) = 0$

If $k \neq m$ then $y(t) = \frac{k}{k-m}\left(e^{-m\,t} - e^{-k\,t}\right) \Longrightarrow \lim_{t\to\infty} y(t) = 0 - 0 = 0$

$z(t) = -\frac{k}{k-m}\,e^{-m\,t} + \frac{m}{k-m}\,e^{-k\,t} + 1 \Longrightarrow \lim_{t\to\infty} z(t) = 0 + 0 + 1 = 1$

8.* Consider the following chemical reaction:

$CH_3\,COO - C_2\,H_5\ +\ Na\,OH \longrightarrow CH_3\,COONa\ +\ C_2\,H_5\,OH$

(ethyl acetate + sodium hydroxide $\longrightarrow$ sodium acetate + ethanol).

The chemical reaction can be written in the form $A + B \xrightarrow{k} X + Y$.

Let $a(t),\ b(t),\ x(t)$ and $y(t)$ respectively denote the concentrations of the species A, B, X and Y at time t where $a(t),\ b(t),\ x(t),\ y(t) \geq 0$ and $k > 0$ is the reaction rate coefficient.

The reaction can be described by the following differential equation system:

(1) $a'(t) = -k\, a(t)\, b(t)$

(2) $b'(t) = -k\, a(t)\, b(t)$

(3) $x'(t) = k\, a(t)\, b(t)$

(4) $y'(t) = k\, a(t)\, b(t)$

Assume that the initial concentrations are $a(0) = a_0 = 0.02$ and $b(0) = b_0 = 0.004$.
If the concentration of ethyl acetate decreases by 10% in 25 minutes then in how many minutes decreases the concentration by 15%?

Solution. (1) + (3) $\Longrightarrow a'(t) + x'(t) = 0$

$\Longrightarrow a(t) + x(t) = \text{constant for all } t \in \mathbb{R}$

If $t = 0$ then $a(0) + x(0) = a_0 + 0$

$\Longrightarrow a(t) + x(t) = a_0 \Longrightarrow a(t) = a_0 - x(t)$

Similarly, (2) + (3) $\Longrightarrow b'(t) + x'(t) = 0$

$\Longrightarrow b(t) + x(t) = \text{constant for all } t \in \mathbb{R}$

If $t = 0$ then $b(0) + x(0) = b_0 + 0$

$\Longrightarrow b(t) + x(t) = b_0 \Longrightarrow b(t) = b_0 - x(t)$

Substituting into (3) : $x'(t) = k\, a(t)\, b(t)$

$$x'(t) = k\,(a_0 - x(t))\,(b_0 - x(t))$$

$$\frac{\text{dx}}{\text{dt}} = k(a_0 - x)\,(b_0 - x)$$

Constant solution: $x \equiv a_0,\ x \equiv b_0$; if $x \neq a_0,\ x \neq b_0$:

$$\int \frac{1}{k(a_0 - x)\,(b_0 - x)}\,\text{dx} = \int \text{dt}$$

Partial fraction decomposition:

$$\Longrightarrow \frac{1}{k(b_0 - a_0)} \int \left(\frac{1}{a_0 - x} - \frac{1}{b_0 - x}\right) \text{dx} = \int \text{dt}$$

$$\Longrightarrow \frac{1}{k(b_0 - a_0)} (-\ln(a_0 - x) + \ln(b_0 - x) + \ln c) = t \Longrightarrow t = \frac{1}{k(b_0 - a_0)} \ln\left(c \cdot \frac{b_0 - x}{a_0 - x}\right)$$

c can be calculated from the initial condition $x(0) = 0$:

$$0 = \frac{1}{k(b_0 - a_0)} \ln\left(c \cdot \frac{b_0}{a_0}\right) \Longrightarrow \ln\left(c \cdot \frac{b_0}{a_0}\right) = 0 \Longrightarrow c \cdot \frac{b_0}{a_0} = 1 \Longrightarrow c = \frac{a_0}{b_0}$$

The general solution is $t = \dfrac{1}{k(b_0 - a_0)} \ln \dfrac{a_0(b_0 - x)}{b_0(a_0 - x)}$

The concentration of A decreases by 10% after 25 minutes:

$t = 25 \Longrightarrow \quad a(25) = 0.9\, a_0$

$a(25) + x(25) = a_0 \Longrightarrow x(25) = 0.1\, a_0 = 0.1 \cdot 0.02 = 0.002$

$$\Longrightarrow\ 25 = \frac{1}{k(0.004 - 0.02)} \ln \frac{0.02\,(0.004 - 0.002)}{0.004\,(0.02 - 0.002)} \ \Longrightarrow\ k = -\frac{1}{25 \cdot 0.016} \ln \frac{40}{72} \approx 1.47$$

The concentration of A decreases by 15% after T minutes:

$a(T) = 0.85\, a_0$

$a(T) + x(T) = a_0 \ \Longrightarrow\ x(T) = 0.15\, a_0 = 0.003$

$$\Longrightarrow\ T = \frac{1}{1.47\,(0.004 - 0.02)} \ln \frac{0.02\,(0.004 - 0.003)}{0.004\,(0.02 - 0.003)} = -\frac{1}{0.0235} \ln \frac{5}{17} \approx 51.4 \text{ minutes}$$