03 - First order linear differential equations, exercises

Solve the following differential equations.

1. In the mixing problem, suppose that the salt content in the inflow decreases exponentially. Then the equation is $y'(t) = 0.6\,e^{-t} - 0.2\,y(t)$

2. $y'(t) = \dfrac{y(t)}{t} + t \qquad (t \neq 0)$

3. $x'(t) + 2\,x(t) = e^t,\ x(0) = 0$

4. $t\,x'(t) - 2\,x(t) = 2\,t^4$

5. $E'(r) = -\dfrac{2}{r}\,E(r) + \dfrac{1}{r}$

(here $E(r) > 0$ is the force field of a point mass and $\dfrac{1}{r}$ is the external force).

6.* The current $I(t)$ in an RC circuit is described by the equation $R\,I'(t) + \dfrac{1}{C}\,I(t) = F(t)$ where $R,\ C > 0$ are constants (R is the resistance and C is the capacity) and $F(t)$ is the external excitation.

a) Find the general solution if $I(0) = I_0$ and there is no external excitation, that is, $F(t) \equiv 0$.

b) Find the general solution if $R = C = 1$ and $F(t) = F_0 \sin t$ is a periodic excitation, where $F_0 > 0$ is a constant. Show that after a long time $I(t)$ can also be considered periodic.

7.* In the chemical reaction $X \xrightarrow{k} Y \xrightarrow{m} Z$, let $x(t),\ y(t)$ and $z(t)$ denote the concentrations of the species X, Y and Z as a function of t, respectively. The reaction is described by the following differential equation system:

$$x'(t) = -k\,x(t)$$
$$y'(t) = k\,x(t) - m\,y(t)$$
$$z'(t) = m\,y(t)$$

where $k > 0$ and $m > 0$ are the reaction rate coefficients.

a) Solve the equation system if $k > m$ and $x(0) = 1,\ y(0) = 0,\ z(t) = 0$.

b) Show that $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} y(t) = 0$ and $\lim\limits_{t\to\infty} z(t) = 1$.

8.* Consider the following chemical reaction: $CH_3COO - C_2H_5 + NaOH \longrightarrow CH_3COONa + C_2H_5OH$ (ethyl acetate + sodium hydroxide $\longrightarrow$ sodium acetate + ethanol).

The chemical reaction can be written in the form $A + B \xrightarrow{k} X + Y$.

Let $a(t),\ b(t),\ x(t)$ and $y(t)$ respectively denote the concentrations of the species A, B, X and Y at time t where $a(t),\ b(t),\ x(t),\ y(t) \geq 0$ and $k > 0$ is the reaction rate coefficient.

The reaction can be described by the following differential equation system:

(1) $a'(t) = -k\, a(t)\, b(t)$
(2) $b'(t) = -k\, a(t)\, b(t)$
(3) $x'(t) = k\, a(t)\, b(t)$
(4) $y'(t) = k\, a(t)\, b(t)$

Assume that the initial concentrations are $a(0) = a_0 = 0.02$ and $b(0) = b_0 = 0.004$.
If the concentration of ethyl acetate decreases by 10% in 25 minutes then in how many minutes decreases the concentration by 15%?