02 - Separable differential equations

Definition

First-order separable differential equations are equations of the form

$$y' = f(x) \cdot g(y)$$

where $f : (a, b) \longrightarrow \mathbb{R}$ and $g : (c, d) \longrightarrow \mathbb{R}$ are continuous functions.
The solution is a function $y = y(x)$ for which

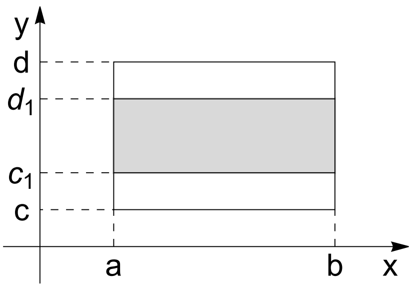
$y'(x) \equiv f(x) \cdot g(y(x)), \quad \forall x \in (a, b)$ (notation: $\forall$ means "for all")

Theorem

The solution of the above separable equation is the following:
1) If $g(y_0) = 0$ (where $y_0 \in (c, d)$) then $y \equiv y_0$ is a solution (since $y' \equiv 0$).
2) If $g(y) \neq 0$ for $y \in (c_1, d_1) \subset (c, d)$ then the initial value problem

$$y' = f(x) \cdot g(y), \quad y(x_0) = y_0, \quad x_0 \in (a, b), \quad y_0 \in (c_1, d_1)$$

has a unique solution $y(x)$ that can be determined by the implicit equation $\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$.



Proof. 1) Substituting $y \equiv y_0$ into $y' = f(x) \cdot g(y)$

$$\Longrightarrow y_0' = f(x) \cdot g(y_0) \Longrightarrow 0 = f(x) \cdot 0 = 0 \Longrightarrow y \equiv y_0 \text{ is a solution.}$$

2) Since $g(y) \neq 0$ then $y' = f(x) \cdot g(y) \iff \frac{y'}{g(y)} = f(x)$.

If $y = y(x)$ $(y(x_0) = y_0)$ is a solution then $\frac{y'(x)}{g(y(x))} = f(x)$

for all $x \in (x_0 - \delta, x_0 + \delta)$ (for some $\delta > 0$).

Since the functions $h(y) = \frac{1}{g(y)}$ and $f(x)$ are continuous, they have antiderivatives on (c_1, d_1) and (a, b), respectively. Let $\frac{dH}{dy} = h(y) = \frac{1}{g(y)}$ and $\frac{dF}{dx} = f(x)$.

Then $\frac{y'(x)}{g(y(x))} = f(x) \iff \frac{d}{dx}(H(y(x))) = \frac{d}{dx}(F(x))$,

so the general solution of the equation is $H(y(x)) = F(x) + C$, $C \in \mathbb{R}$.

From the initial condition $y(x_0) = x_0$, the constant C can be determined uniquely:

$H(y(x_0)) = F(x_0) + C \implies C = H(y(x_0)) - F(x_0)$

Conversely, assume that $H(y(x)) = F(x) + C$ for some constant C.

Then differentiating both sides with respect to x, we get that

$h(y(x))\, y'(x) = f(x) \implies \frac{y'(x)}{g(y(x))} = f(x) \implies y(x)$ is a solution of $y' = f(x) \cdot g(y)$.

Remark. Integrating both sides with respect to x: $\frac{y'(x)}{g(y(x))} = f(x) \implies \int \frac{y'(x)}{g(y(x))}\, dx = \int f(x)\, dx$

By the substitution formula $\left(y'(x)\, dx = \frac{dy}{dx}\, dx = dy\right) \implies \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$

Summary

$y'(x) = f(x)\, g(y(x))$

1st step: Finding the constant solutions (if there are any).

If $y(x) \equiv c$ then $y'(x) \equiv 0$ so from the equation $f(x)\, g(y(x)) \equiv 0 \implies g(y(x)) \equiv 0$

and the constant solution(s) $y(x) \equiv c$ can be expressed.

2nd step: We rewrite the derivative as $y'(x) = \frac{dy}{dx}$ and then handle y as a variable.

Assuming that $g(y) \neq 0$ on an interval I and dividing by $g(y)$:

$\frac{dy}{dx} = f(x)\, g(y) \iff \frac{dy}{g(y)} = f(x)\, dx \iff \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$

Integrating both sides, we get an equation of the form $G(y) = F(x) + c$ ($c \in \mathbb{R}$) from where we have to express y as a function of x.

Special cases

a) If $g(y) = 1$ then $y'(x) = f(x)$ is a **directly integrable equation**

b) If $f(x) = 1$ then $y' = g(y)$ is an **autonomous equation**

(the right-hand side doesn't depend explicitly on x)

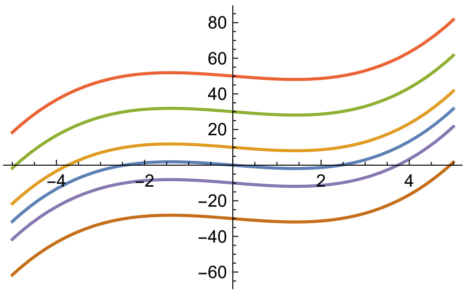
Remark

a) If $y'(x) = f(x)$ then the solutions are $y(x) = F(x) + C$, where $F'(x) = f(x)$ and $C \in \mathbb{R}$.
$\Longrightarrow$ the graphs of the solutions are shifted vertically

b) If $y(x)$ is a solution of the autonomous equation $y' = g(y)$ then for all $s \in \mathbb{R}$ the function $z(x) = y(x+s)$ is also a solution since $z'(x) = y'(x+s) = g(y(x+s)) = g(z(x))$.
$\Longrightarrow$ the graphs of the solutions are shifted horizontally

It means that is enough to determine the solutions at $x = 0$ since the other solutions can be obtained by shifting them horizontally.

a)



b)