02 - First-order separable differential equations, solutions

1.

Solve the following differential equations:

a) $y'(x) = \dfrac{y^2(x)}{x^2}$ b) $y'(x) = \dfrac{x^2}{y^3(x)}$ c) $x'(t) = \dfrac{x(t)}{t},\ x(1) = 2$

d) $y'(x) = \dfrac{2\,y(x)}{x}$ e) $x'(t) = \dfrac{x(t)}{t^2},\ x(1) = 1$ f) $x'(t) = \dfrac{t}{x(t)},\ x(1) = -2$

Solutions and results

a) $y'(x) = \dfrac{y^2(x)}{x^2}$ $(x \neq 0)$

Since $x \neq 0$ then the domain of the solution is an interval that doesn't contain 0.

Constant solution: $y(x) \equiv c \implies y'(x) \equiv 0 \implies \dfrac{y^2(x)}{x^2} \equiv 0 \implies y(x) \equiv 0$

(that is, $y(x) = 0$ if $x > 0$ or $y(x) = 0$ if $x < 0$)

If $y \neq 0$ then separating the variables: $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{y^2}{x^2} \implies \displaystyle\int \frac{1}{y^2}\,\mathrm{dy} = \int \frac{1}{x^2}\,\mathrm{dx}$

$\implies -\dfrac{1}{y} = -\dfrac{1}{x} + c = \dfrac{-1 + c\,x}{x}$ (general solution in implicit form)

$\implies y = \dfrac{x}{1 - c\,x}$ (general solution in explicit form)

All solutions: $y = \dfrac{x}{1 - c\,x}$ or $y \equiv 0$

b) $y'(x) = \dfrac{x^2}{y^3(x)}$ $(y(x) \neq 0)$

If $y(x) \equiv c \implies y'(x) \equiv 0 \implies \dfrac{x^2}{y^3(x)} = 0 \implies x = 0$ but this is not a constant solution,

only a point in the domain of the solution. There is no constant solution here.

If $x \neq 0$ then separating the variables: $\dfrac{\mathrm{dy}}{\mathrm{dx}} = \dfrac{x^2}{y^3} \implies \displaystyle\int y^3\,\mathrm{dy} = \int x^2\,\mathrm{dx} \implies \frac{y^4}{4} = \frac{x^3}{3} + c \implies y(x) = \pm\sqrt[4]{\frac{4}{3}x^3 + C}$

c) $x'(t) = \dfrac{x(t)}{t},\ x(1) = 2$ $(t \neq 0)$

Constant solution: $x(t) \equiv 0$ (that is, $x(t) = 0$ if $t > 0$ or $x(t) = 0$ if $t < 0$)

If $x \neq 0$ then separating the variables: $\dfrac{\mathrm{dx}}{\mathrm{dt}} = \dfrac{x}{t} \implies \displaystyle\int \frac{1}{x}\,\mathrm{dx} = \int \frac{1}{t}\,\mathrm{dt}$

$\implies \ln|x| = \ln|t| + c \quad (c \in \mathbb{R})$

$\implies |x| = e^{\ln|t| + c} = e^{\ln|t|} \cdot e^c = |t| \cdot e^c$

$\implies x = \pm e^c \cdot t$ or $x \equiv 0 \quad (c \in \mathbb{R})$

Here e^c can be any positive number and $\pm e^c$ can be any positive and any negative number.

We can summarize the two solutions (when $C = 0$ then we get the constant solution), so the general solution is:

$x(t) = C \cdot t$

where the domain of $x(t)$ is $t > 0$ or $t < 0$.
The initial condition: $x(1) = 2 \implies 2 = C \cdot 1 \implies C = 2$
The solution of the initial value problem: $x(t) = 2t$
Since $x(t)$ is defined at $t = 1$ then we may suppose that the domain of the solution is $t > 0$.

d) $y'(x) = \dfrac{2y(x)}{x} \quad (x \neq 0)$

General solution: $y(x) = C x^2 \; (C \in \mathbb{R})$

e) $x'(t) = \dfrac{x(t)}{t^2}, \; x(1) = 1 \quad (t \neq 0)$

General solution: $x(t) = C e^{-\frac{1}{t}} \; (C \in \mathbb{R})$
Solution of the initial value problem: $x(t) = e^{1-\frac{1}{t}}$

f) $x'(t) = \dfrac{t}{x(t)}, \; x(1) = -2 \quad (x(t) \neq 0)$

General solution: $x(t) = \pm\sqrt{t^2 + C}$
Solution of the initial value problem: $x(t) = -\sqrt{t^2 + 3}$

2.*

Solve the differential equation $x'(t) = a^2 - x^2(t)$ where $a = 2$ and
a) $x(0) = 1$ b) $x(0) = 3$

Solution. $\dfrac{dx}{dt} = 4 - x^2$
Constant solutions: $4 - x^2(t) = 0 \implies x(t) \equiv 2$ or $x(t) \equiv -2 \; (x \in \mathbb{R})$.
If $x \neq 2$ or $x \neq -2$ then by separating the variables,
$\int \dfrac{1}{4 - x^2}\, dx = \int dt \implies$ partial fraction decomposition:

$$\frac{1}{4 - x^2} = \frac{-1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$$

$\implies \quad -1 = A(x+2) + B(x-2)$

$x = -2 \implies -1 = 0 + B(-4) \implies B = \dfrac{1}{4}$

$x = 2 \implies -1 = A \cdot 4 + 0 \implies A = -\dfrac{1}{4}$

The partial fraction decomposition: $\dfrac{1}{4 - x^2} = -\dfrac{1}{4}\dfrac{1}{x-2} + \dfrac{1}{4}\dfrac{1}{x+2}$

$$\Longrightarrow \int \frac{1}{4}\left(\frac{1}{x+2} - \frac{1}{x-2}\right) dx = \int dt$$

$$\Longrightarrow \frac{1}{4} \ln |x+2| - \frac{1}{4} \ln |x-2| = t + c_1$$ (the general solution in implicit form)

a) From the initial condition $x(0) = 1$ it may be supposed that $-2 < x < 2 \Longrightarrow x + 2 > 0$ and $x - 2 < 0$.

$$\frac{1}{4} \ln(x+2) - \frac{1}{4} \ln(2-x) = t + c_1 \Longrightarrow \ln \frac{x+2}{2-x} = 4t + 4c_1 \Longrightarrow \frac{x+2}{2-x} = e^{4t+4c_1} = :A$$

$$\Longrightarrow x + 2 = 2A - Ax \Longrightarrow x(1+A) = 2A - 2 \Longrightarrow x = \frac{2A-2}{1+A} = \frac{2e^{4t+4c_1} - 2}{1 + e^{4t+4c_1}} = \frac{2e^{4t} \cdot e^{4c_1} - 2}{1 + e^{4t} \cdot e^{4c_1}}$$

The general solution is $x(t) = \frac{2Ce^{4t} - 2}{1 + Ce^{4t}}$

(All solutions: $x(t) = \frac{2Ce^{4t} - 2}{1 + Ce^{4t}}$, $x(t) \equiv 2$, $x(t) \equiv -2$)

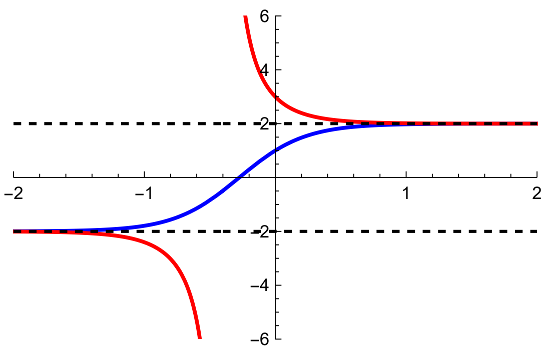
$$x(0) = 1 \;\; (x = 1,\; t = 0) \Longrightarrow 1 = \frac{2C \cdot e^0 - 2}{1 + C \cdot e^0} = \frac{2C - 2}{1 + C} \Longrightarrow C = 3$$

The solution of the initial value problem is $x(t) = \frac{6e^{4t} - 2}{1 + 3e^{4t}}$

b) Similarly, if $x(0) = 3$ then $x + 2 > 0$ and $x - 2 > 0 \Longrightarrow \frac{1}{4} \ln(x+2) - \frac{1}{4} \ln(x-2) = t + c_1$

The solution of the initial value problem is $x(t) = \frac{10e^{4t} + 2}{-1 + 5e^{4t}}$

Plotting the solutions with case a) in blue and b) in red:



3. Autocatalytic reaction

First-order differential equations need not have solutions that are defined for all times.

a) Find the general solution of the equation $x'(t) = x(t)^2$.

b) Solve the initial value problems $x(0) = 1$ and $x(2) = -1$.

c) Discuss the domains over which each solution is defined.

Remark: The equation (where $x(t) \geq 0$ and $t \geq 0$) can be considered as the model of the autocatalytic reaction $2X + Y \rightarrow 3X$ where $x(t)$ denotes the concentration of the substance X and the concentration of Y is considered constant. Show that in this case the solution "blows up" in a finite time, that is, for all solution x, there exists a $T > 0$ such that $\lim_{t \to T-} x(t) = +\infty$.

Solution.

a) $x'(t) = x^2(t)$ or $\dfrac{dx}{dt} = x^2$

Constant solution: if $x(t) \equiv c \implies x'(t) \equiv 0 \implies x(t) \equiv 0,\ t \in \mathbb{R}$.

If $x \neq 0$ then by separating the variables: $\int \frac{1}{x^2}\, dx = \int dt \implies -\frac{1}{x} = t + c$

The general solution is $x(t) = -\dfrac{1}{t+c}$ ($t \in I$ where I is a suitable interval)

or $x(t) \equiv 0,\ t \in \mathbb{R}$.

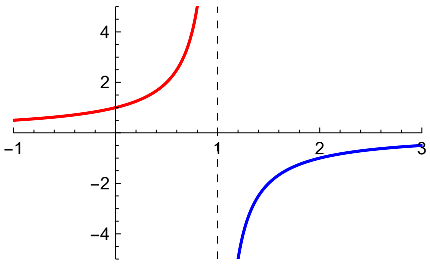
b) Initial value problems:

(1) $x(0) = 1 \implies 1 = -\dfrac{1}{0+c} \implies c = -1 \implies x(t) = \dfrac{-1}{t-1}$.

Since $t \neq 1$ and $x(0) > 0$ then from $x(t) > 0$ it may be supposed that the domain of the solution is the interval $t < 1$, see the red graph.

(2) $x(2) = -1 \implies -1 = -\dfrac{1}{2+c} \implies c = -1 \implies x(t) = \dfrac{-1}{t-1}$.

Since $t \neq 1$ and $x(2) < 0$ then from $x(t) < 0$ it may be supposed that the domain of the solution is the interval $t > 1$, see the blue graph.



Observation: The domain of the solution depends on the initial condition.

Remark: If $x(t) \geq 0,\ t \geq 0$ and $x(0) = x_0 > 0$ then the value of x becomes infinitely large in finite time (this is called a "blow up"). Substituting the initial condition $x(0) = x_0$ into the general solution

$$x(t) = -\frac{1}{t+c} \implies x_0 = -\frac{1}{c} \implies c = -\frac{1}{x_0} < 0$$

Since $x(t) \geq 0$ then $t + c < 0 \implies 0 \leq t < -c = \dfrac{1}{x_0} =: T\ (T > 0)$.

The general solution is $x(t) = -\dfrac{1}{t-T} \implies \lim_{t \to T-0} x(t) = +\infty$ (see the red graph in the figure).

4. Radioactivity, exponential decay

Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. If $N(t)$ denotes the amount of radioactive material at time t then this process can be expressed by the differential equation

$$N'(t) = -\lambda N(t)$$

where $\lambda > 0$ is the exponential decay constant.

a) Find the general solution.

b) Show that there exists a number $T > 0$ (T is called the half-life) such that $N(t+T) = \frac{1}{2} N(t)$ for all $t > 0$. Express T by λ.

c) If the time is measured in hours, $\lambda = 10^{-3}$, the initial mass is $N(0) = 271$ kg then what will be the value of $N(t)$ after 1000 hours?

d) The the half-life of the radium is 1600 years. What percent of the initial quantity decays after 100 years?

e) A rock contains 100 mg of uranium and 14 mg of lead. The half-life of the uranium is $4.5 \cdot 10^9$ years and during the decay of 238 g of uranium, 206 g of lead is produced. How old is the rock?

Solution.

a) $N'(t) = -\lambda N(t)$ $(\lambda > 0)$ or $\frac{dN}{dt} = -\lambda N$ (autonomous equation)

Constant solution: $N(t) \equiv 0$. If $N \neq 0$ then separating the variables:

$\int \frac{1}{N} \mathrm{dN} = -\int \lambda \, \mathrm{dt} \implies \ln |N| = -\lambda t + c_1$ $(c_1 \in \mathbb{R})$

$\implies |N| = e^{-\lambda t + c_1} = e^{-\lambda t} \cdot e^{c_1} \implies N = \pm e^{c_1} \cdot e^{-\lambda t}$ or $N \equiv 0$.

The general solution: $N(t) = C \cdot e^{-\lambda t}$, $C \in \mathbb{R}$ (here $N(t) \geq 0$, so $C \geq 0$).

Initial condition: $N(0) = N_0$ (the initial quantity) $\implies N_0 = C \cdot e^0 \implies C = N_0$

The solution of the initial value problem: $N(t) = N_0 \cdot e^{-\lambda t}$

b) $N(t+T) = \frac{1}{2} N(t) \implies N_0 e^{-\lambda(t+T)} = \frac{1}{2} N_0 e^{-\lambda t} \implies e^{-\lambda T} = \frac{1}{2} \implies -\lambda T = \ln\left(\frac{1}{2}\right) = -\ln 2 \implies T = \frac{\ln 2}{\lambda}$

c) $N(t) = N(0) e^{-\lambda t} \implies N(1000) = 271 e^{-1} \approx 99.7$ kg

d) $T = 1600 \implies \lambda = \frac{\ln 2}{1600} \approx 0.000433$

$N(t) = N_0 e^{-\lambda t} \implies N(100) = N_0 e^{-\frac{\ln 2}{1600} \cdot 100} \implies \frac{N(100)}{N_0} = e^{-0.0433} = 0.958$

$\implies$ 100 % − 95.8 % = 4.2 % decays in 100 years.

e) Let $N(t)$ denote the amount of uranium in the rock. From the half-life λ can be calculated:

$N(4.5 \cdot 10^9) = \frac{N(0)}{2} = N(0) e^{-\lambda \cdot 4.5 \cdot 10^9}$, from where $\lambda = \frac{\ln 2}{4.5} \cdot 10^{-9} \approx 0.154 \cdot 10^{-9}$

The initial amount of uranium in the rock is

$N(0) = 100 + \frac{238}{206} \cdot 14 = 116.175$ mg, so $N(t) = 116.175 \, e^{-0.154 \cdot 10^{-9} t}$

At time T, the rock contains 100 mg of uranium: $100 = 116.175 \, e^{-0.154 \cdot 10^{-9} T}$

$\implies T = 9.74 \cdot 10^8$, so the age of the rock is approximately 974 million years.

5. Unlimited growth of bacteria

The unlimited growth of bacteria can be modelled with the equation $y'(t) = K y(t)$ where $K > 0$ and $y(t) \geq 0$ denotes the amount of bacteria at time t.

a) Find the general solution.

b) Find the solution if $y(0) = y_0$.

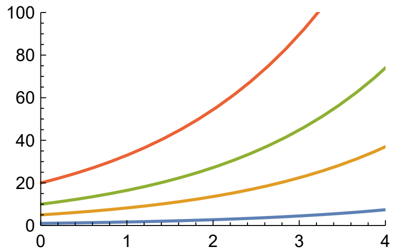
Solution. See exercise 4. a)

a) The general solution is $y(t) = C\,e^{K\,t}, \quad C > 0$

b) If $y(0) = y_0$ then the solution is $y(t) = y_0\,e^{K\,t}$

$\Longrightarrow \lim\limits_{t\to+\infty} y(t) = +\infty$, since $y_0 > 0$ and $K > 0$.

Plotting a few solutions:



Remark. Consider the bacterium *Escherichia coli*. The mass of one bacterium is $2\cdot 10^{-12}$ g.

Its life cycle is about thirty minutes, so that is the doubling time.

It means that if t is measured in minutes then

$$\frac{y(t+30)}{y(t)} = \frac{y_0\,e^{K\,(t+30)}}{y_0\,e^{K\,t}} = e^{30\,K} = 2 \Longrightarrow K = \frac{\ln 2}{30} = 0.0231049$$

After 3 days, that is, after $t = 72\cdot 60 = 4320$ minutes the total mass of bacteria is

$$y(4320) = 2\cdot 10^{-12}\,e^{\frac{\ln 2}{30}\cdot 4320} = 2\cdot 10^{-12}\cdot 2^{\frac{4320}{30}} = 2\cdot 10^{-12}\cdot 2^{144} = 4.46015\cdot 10^{31}\,g$$

As a comparison the mass of the Earth is $5.972\cdot 10^{27}$ g.

It means that the model is not valid for such a long period of time, since there is no longer enough food for the bacteria. The logistic population model is a more accurate model, see exercise 8.

6. Heating problem

Newton’s law of cooling: The rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of its surroundings.

The process can be modelled by the equation

$$T'(t) = K(T_A - T(t))$$

where $K > 0$, $T_A > 0$ are constants. $T(t)$ denotes the temperature of the object at time t and T_A denotes the temperature of the surrounding medium.

a) When taken out of the oven, the temperature of a loaf decreases from 100 °C to 60 °C in 10 minutes. The temperature of the surrounding air is 20 °C. At what time will the temperature of the loaf reach 25 °C?

b) If the initial temperature of the loaf is 120 °C, the temperature of the air is $T_A = 30$ °C and $K = 0.0366$ then what will be the temperature of the loaf after 60 minutes?

Solution.

a) $T'(t) = K(T_A - T(t)) \quad (K > 0)$ or

$$\frac{dT}{dt} = K(T_A - T) \text{ (autonomous equation)}$$

Observation: if $T > T_A \implies T' < 0 \implies T$ decreases (cooling)
if $T < T_A \implies T' > 0 \implies T$ increases (heating)

Constant solution: $T(t) \equiv T_A$. If $T \not\equiv T_A$ then by separating the variables,

$$\int \frac{1}{T_A - T} \, dT = \int K \, dt \implies -\ln|T_A - T| = K\,t + c_1 \implies |T_A - T| = e^{-K\,t - c_1}$$

$$\implies T_A - T = \pm e^{-K\,t} \cdot e^{-c_1} \implies T = T_A \pm e^{-c_1} \cdot e^{-K\,t} \text{ or } T \equiv T_A$$

The general solution is $T(t) = T_A + C \cdot e^{-K\,t}$, $C \in \mathbb{R}$

From the data: $T_A = 20$, $T(0) = 100 = 20 + C$ $\implies C = 80$, $K = \frac{\ln 2}{10}$
$T(10) = 60 = 20 + C\,e^{-10\,K}$

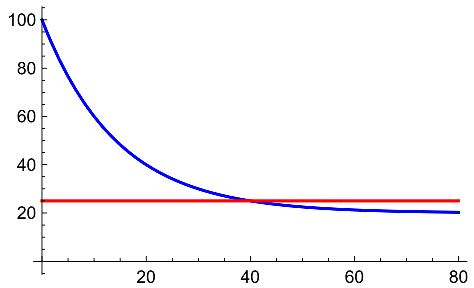
The solution of the initial value problem with the given data: $T(t) = 20 + 80\,e^{-0.0693\,t}$

Let t_* denote the time when the temperature of the loaf reaches 25 °C:

$$T(t_*) = 25 = 20 + 80\,e^{-k\,t_*} \implies t_* = 10\,\frac{\ln 16}{\ln 2} = 40 \text{ (minutes)}$$

The solution can be seen in Figure 1.

Figure 2. shows a few solutions with initial temperatures 100 °C, 60 °C, 20 °C, 0 °C, –20 °C.

1.


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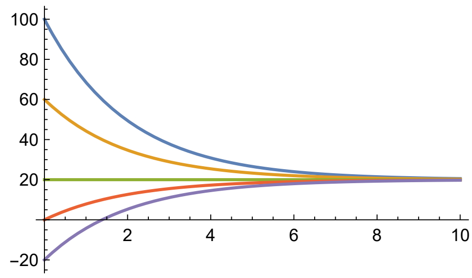


b) $T(0) = 120 = 30 + C \cdot e^{-0.0366 \cdot 0} \implies C = 90 \implies T(60) = 30 + 90 \cdot e^{-0.0366 \cdot 60} \approx 40.$

7. Mixing problem

Suppose a brine containing 0.3 kg of salt per liter runs into a tank initially filled with 10 liters of water. The brine runs into the tank at the rate of 2 liter/min, the mixture is kept uniform by stirring, and the mixture flows out at the same rate. The equation giving the amount of salt in the tank at time t is $y'(t) = 0.6 - 0.2\,y(t)$.

a) Find the general solution. Show that the general solution converges to the constant solution.

b) How much salt is in the tank in 5 minutes?

Solution.

Let $y(t)$ denote the amount of chemical (salt) in the container and let $V(t)$ denote the total volume of liquid in the container at time t.

Model: $\begin{pmatrix}\text{rate of change of the amount of}\\ \text{chemical in the container}\end{pmatrix}$ = (chemical's arrival rate) – (chemical's departure rate)

$$\text{arrival rate} = \left(0.3\,\frac{\text{kg}}{\text{liter}}\right)\left(2\,\frac{\text{liter}}{\text{min}}\right) = 0.6\,\frac{\text{kg}}{\text{min}}$$

$$\text{departure rate} = \frac{y(t)}{V(t)} \times \text{outflow rate} = \left(\frac{y(t)}{10}\,\frac{\text{kg}}{\text{liter}}\right)\left(2\,\frac{\text{liter}}{\text{min}}\right)$$

From the model we obtain the following differential equation:

$y'(t) = 0.6 - 0.2\,y(t) = 0.2\,(3 - y(t))$ or $\frac{dy}{dt} = 0.2\,(3 - y)$ (autonomous equation)

Constant solution: $y(t) \equiv 3$. If $y \neq 3$ then by separating the variables,

$$\int \frac{1}{3-y}\,\mathrm{dy} = \int 0.2\,\mathrm{dt} \implies -\ln|3-y| = 0.2\,t + c_1 \implies |3-y| = e^{-0.2\,t - c_1}$$

$$\implies 3 - y = \pm e^{-0.2\,t} \cdot e^{-c_1} \implies y = 3 \pm e^{-c_1} \cdot e^{-0.2\,t} \text{ or } y \equiv 3$$

The general solution is $y(t) = 3 + C \cdot e^{-0.2\,t}$, $C \in \mathbb{R}$

$\implies \lim\limits_{t \to +\infty} y(t) = 3 + 0 = 3$, that is, the general solution tends to the constant solution if $t \to +\infty$.

b) From the initial condition: $y(0) = 0 = 3 + C \implies C = -3 \implies y(t) = 3\left(1 - e^{-0.2\,t}\right)$

After 5 minutes the amount of salt in the container is $y(5) = 3\left(1 - e^{-1}\right) \approx 1.896$ kg

8*. The logistic population model

The limited growth of bacteria can be modelled by the equation $y'(t) = K\,y(t)\,(M - y(t))$ where $K > 0$, $M > 0$ and M is the carrying capacity of the environment.
Find the general solutions $y(t) \geq 0$ and sketch the graph of the solutions.

Solution.

$y'(t) = K\,y(t)\,(M - y(t))$ or $\frac{dy}{dt} = K\,y(M - y)$

Observation: if $y > M \implies y' < 0 \implies y$ decreases
if $0 \leq y < M \implies y' > 0 \implies y$ increases

Constant solutions: $y(t) \equiv 0$ or $y(t) \equiv M$. If $y \neq 0$, $y \neq M$ then by separating the variables,

$$\int \frac{1}{y(M-y)}\,\mathrm{dy} = \int K\mathrm{dt} \implies \text{Partial fraction decomposition:}$$

$$\int\left(\frac{1}{M}\cdot\frac{1}{y} + \frac{1}{M}\cdot\frac{1}{(M-y)}\right)\mathrm{dy} = \int K\mathrm{dt} \implies \frac{1}{M}\ln|y| - \frac{1}{M}\ln|M-y| = K\,t + c_1$$

$$\implies \frac{1}{M}\ln\left|\frac{y}{M-y}\right| = K\,t + c_1 \implies \left|\frac{y}{M-y}\right| = e^{M\,K\,t + M\,c_1}$$

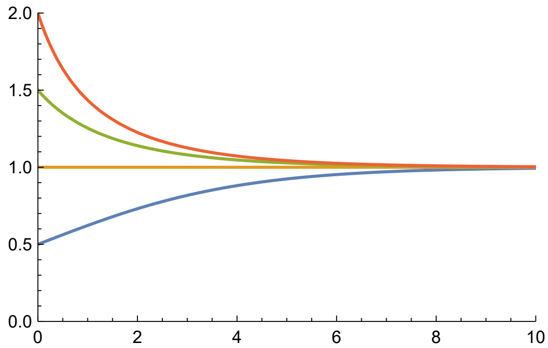
$$\implies \frac{y}{M-y} = \pm e^{M\,K\,t + M\,c_1} = C\,e^{M\,K\,t} =: A$$

$\Longrightarrow\ y = A(M - y) \Longrightarrow y(1 + A) = A M \Longrightarrow y = \dfrac{A M}{1 + A} = \dfrac{M C e^{M K t}}{1 + C e^{M K t}}$

The general solution is $y(t) = \dfrac{M C e^{M K t}}{1 + C e^{M K t}}$, $C \in \mathbb{R}$ (it contains the constant zero solution if $C = 0$)

Remark: if $C \neq 0$ then dividing by $C e^{M K t}$, we get $y(t) = \dfrac{M}{1 + D e^{-M K t}} \Longrightarrow \lim\limits_{t \to \infty} y(t) = \dfrac{M}{1 + 0} = M$.

Plotting a few solutions with $K = 0.5$, $M = 1$:



9*. Parachutist

The equation $m v'(t) = m g - k v^2(t)$ describes the motion of a parachutist of mass m where $v(t)$ is the velocity at time t and $m v'(t)$ is mass times acceleration. By Newton's second law, $m v'(t)$ equals the two forces, the attraction $m g$ by the earth and $-k v^2(t)$, the air resistance.

a) Find the general solution. Denoting the constant solutions by b, the equation can be transformed into the form

$v'(t) = -\dfrac{k}{m} (v^2(t) - b^2)$. $\left(b = \sqrt{\dfrac{m g}{k}}\right)$

b) Show that $v(t) \to b$ if $t \to +\infty$.

Solution. a) $\dfrac{d v}{d t} = -\dfrac{k}{m} (v^2 - b^2)$

Constant solutions: $v(t) \equiv \pm b$. If $v \neq \pm b$ then by separating the variables,

$\displaystyle\int \frac{1}{v^2 - b^2}\, dv = -\frac{k}{m} \int dt \Longrightarrow$

Partial fraction decomposition: $\displaystyle\int \left(\frac{1}{2 b} \cdot \frac{1}{v - b} - \frac{1}{2 b} \cdot \frac{1}{v + b}\right) dv = -\frac{k}{m} \int dt \Longrightarrow$

$\dfrac{1}{2 b} \ln |v - b| - \dfrac{1}{2 b} \ln |v + b| = -\dfrac{k}{m} t + c_1 \Longrightarrow \dfrac{1}{2 b} \ln \left|\dfrac{v - b}{v + b}\right| = -\dfrac{k}{m} t + c_1 \Longrightarrow$

$\left|\dfrac{v - b}{v + b}\right| = e^{-\frac{k}{m} \cdot 2 b \cdot t + 2 b \cdot c_1} \Longrightarrow \dfrac{v - b}{v + b} = \pm e^{-\frac{k}{m} \cdot 2 b \cdot t + 2 b \cdot c_1} = C e^{-\frac{k}{m} \cdot 2 b \cdot t} =: A \Longrightarrow$

$v - b = A(v + b) \Longrightarrow v(1 - A) = b(1 + A) \Longrightarrow v = \dfrac{b(1 + A)}{1 - A}$

The general solution is $v(t) = b\,\dfrac{1 + C\,e^{-\frac{k}{m}\cdot 2\,b\cdot t}}{1 - C\,e^{-\frac{k}{m}\cdot 2\,b\cdot t}}$, $C \in \mathbb{R}$

b) If $t \to +\infty$ then the exponential terms tend to 0, so $\lim\limits_{t\to+\infty} v(t) = b$.

10. Fish population

A fish population in a lake can be modelled by the equation $x'(t) = K\,x(t) - H$ where $K > 0$ is the growth rate and $H > 0$ is the harvest quota. Let $x(0) = x_0 > 0$ be the initial population. Determine the values H of the harvest quota for which the population will not become extinct, that is, $x(t)$ is greater than a positive bound for all $t > 0$.

Solution.

$\dfrac{\mathrm{dx}}{\mathrm{dt}} = K\,x - H$, where $K > 0$, $H > 0$, $x(t) \geq 0$

Constant solution: $x'(t) = 0 \implies x(t) = \dfrac{H}{K}$ for all $t \geq 0$.

If $x(t) \neq 0$ then by separating the variables we get:

$$\int \frac{1}{K\,x - H}\,\mathrm{dx} = \int \mathrm{dt} \implies \frac{\ln|K\,x - H|}{K} = t + c_1 \implies |K\,x - H| = e^{K\,t + c_1} \implies K\,x - H = \pm e^{c_1}\cdot e^{K\,t} \implies$$

$$x = \frac{H}{K} \pm \frac{e^{c_1}}{K}\cdot e^{K\,t} \text{ or } x \equiv \frac{H}{K} \implies \text{The general solution is } x(t) = \frac{H}{K} + C\cdot e^{K\,t} \text{ where } C \in \mathbb{R}.$$

From the initial condition $x(0) = x_0$ we have $x_0 = \dfrac{H}{K} + C\cdot 1 \implies C = x_0 - \dfrac{H}{K}$,

so the solution of the initial value problem is $x(t) = \dfrac{H}{K} + \left(x_0 - \dfrac{H}{K}\right) e^{K\,t}$.

What happens if $t \to \infty$? Since $\lim\limits_{t\to\infty} e^{K\,t} = \infty$ $(K > 0)$ then there are three possibilities.

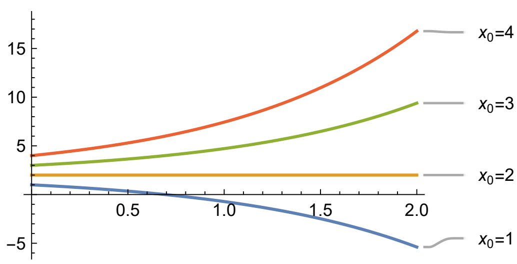
1) If $C > 0$, that is, $H < K\,x_0$ then $\lim\limits_{t\to\infty} x(t) = \infty$.

2) If $C = 0$, that is, $H = K\,x_0$ then $\lim\limits_{t\to\infty} x(t) = \dfrac{H}{K}$.

3) If $C < 0$, that is, $H > K\,x_0$ then $x(t)$ will reach 0 in finite time since $x(0) > 0$ and $\lim\limits_{t\to\infty} x(t) = -\infty$.

The population will not become extinct if $H \leq K\,x_0$.

Plotting a few solutions with $K = 1$, $H = 2$:



11. Outflow from a tank. Torricelli's law.

Torricelli's law states that water issues from a hole in the bottom of a tank with velocity $v(t) = 0.6\,\sqrt{2\,g\,h(t)}$ where $h(t)$ is the height of the water above the hole at time t and g is the acceleration of gravity. The outflow can be modelled by the equation

$h'(t) = -A\sqrt{h(t)}$ where $A = \frac{r^2\pi}{R^2\pi} \cdot 0.6 \cdot \sqrt{2g}$.

When will the tank be empty if $R = 0.9\,m$, $r = 3$ cm and $h(0) = 2.45\,m$?

Remark: If the flow is frictionless then $v(t) = \sqrt{2g\,h(t)}$. However, due to friction, the actual speed is lower. In the case of a circular opening, experience shows that $v(t) = 0.6\sqrt{2g\,h(t)}$.

Solution.

$A = \frac{r^2\pi}{R^2\pi} \cdot 0.6 \cdot \sqrt{2g} = \frac{0.03^2}{0.9^2} \cdot 0.6 \cdot \sqrt{2 \cdot 9.81} \approx 0.00295296$

$h'(t) = -A\sqrt{h(t)}$ or $\frac{dh}{dt} = -A\sqrt{h}$ (autonomous equation)

Constant solution: $h(t) \equiv 0$. If $h \neq 0$ then by separating the variables,

$\int \frac{1}{\sqrt{h}}\,\mathrm{dh} = -\int A\,\mathrm{dt} \implies 2\sqrt{h} = -A\,t + c$

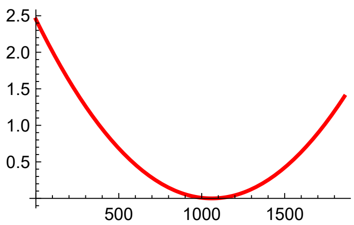
The general solution is $h(t) = \frac{1}{4}(A\,t - c)^2$

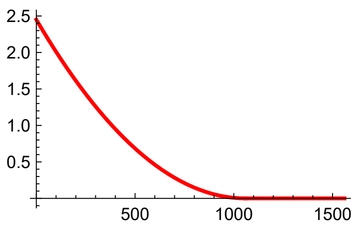
From the initial condition: $h(0) = \frac{1}{4}c^2 = 2.45 \implies c = 3.1305$

The tank will be empty at time t if $h(t) = 0 \implies t = \frac{c}{A} \approx 1060.12\,s \approx 17.67$ min

Remark.

In the equation $h' = -A\sqrt{h}$, the right-hand side is not Lipschitz-continuous at $h = 0$, so the solution is not unique. The general solution is a parabola, however, the water level remains zero when it reaches zero.





12. Current in a closed RL circuit

The switch in an RL circuit is closed at time $t = 0$. The current I obeys the differential equation $L\,I'(t) + R\,I(t) = V$.

a) Find the solution if $I(0) = 0$.

b) How many seconds after the switch is closed will it take the current I to reach half of its steady state value?

Solution.

a) $L\,I'(t) + R\,I(t) = V \implies \frac{dI}{dt} = \frac{1}{L}(V - R\,I)$ (autonomous equation)

See the solution of exercises 5 and 6.

The general solution is $I(t) = \frac{V}{R} + C e^{-\frac{R t}{L}}$

The solution of the initial value problem $I(0) = 0$ is $I(t) = \frac{V}{R}\left(1 - e^{-\frac{R t}{L}}\right)$

b) The steady state value is the constant solution $I \equiv \frac{V}{R}$.

From the equation $\frac{V}{R}\left(1 - e^{-\frac{R t}{L}}\right) = \frac{V}{2R} \implies t = \frac{L \ln 2}{R}$.

13*. Snowballs

Two snowballs are thrown into the room. The radius of the bigger one is double the radius of the smaller one. The melting rate is proportional to the surface area. Find the radius of the bigger snowball at the time when the smaller one is completely melted.

Solution.

We may suppose that the shape of the snowballs is a sphere. Let $r(t)$ denote the radius of either of the snowballs at time t. Then the volume and surface area of the snowball at time t is

$$V(r(t)) = \frac{4\pi}{3} r^3(t), \quad A(r(t)) = 4\pi r^2(t)$$

Since the snowball is melting then its mass will decrease, so

$$\frac{d}{dt} m(t) = \frac{d}{dt}(\rho V(r(t))) = -k A(r(t)),$$

where $\rho > 0$ is the density of the snowball and $k > 0$ is a constant that describes the speed of the melting. From here, applying the chain rule, we get

$$4\pi r^2(t) r'(t) = -\frac{k}{\rho} 4\pi r^2(t)$$

$$r'(t) = -K,$$

where $K = \frac{k}{\rho} > 0$ is a constant.

The general solution of the differential equation is $r(t) = -K t + c$ where c is an arbitrary real number. Suppose that the two snowballs were thrown into the room at time $t = 0$. From here we get that $r(0) = c$ and it can be seen that according to this model the radius of the snowball linearly decreases as t increases. If the initial radius of the smaller snowball is R and the initial radius of the bigger snowball is $2R$ then their radii are described by the following functions:

$$r_1(t) = -K t + R$$

$$r_2(t) = -K t + 2R$$

Let T denote the time when the smaller snowball is completely melted. Then

$r_1(T) = -K\,T + R = 0 \implies T = \frac{R}{K}$

At this time the radius of the bigger snowball is $r_2(T) = -K\,T + 2\,R = -K\,\frac{R}{K} + 2\,R = R$,

that is, it is the same as the initial radius of the smaller one.

14*. The logistic model with constant harvesting

Find the general solution of the logistic differential equation with constant harvesting
$x'(t) = x(t)\,(1 - x(t)) - h$
for all values of the parameter $h > 0$.

Solution.
See chapter 1.3 from the following book:
http://www.math.upatras.gr/~bountis/files/def-eq.pdf