02 - First-order separable differential equations, exercises

1.

Solve the following differential equations:

a) $y'(x) = \dfrac{y^2(x)}{x^2}$ b) $y'(x) = \dfrac{x^2}{y^3(x)}$ c) $x'(t) = \dfrac{x(t)}{t},\ x(1) = 2$

d) $y'(x) = \dfrac{2\,y(x)}{x}$ e) $x'(t) = \dfrac{x(t)}{t^2},\ x(1) = 1$ f) $x'(t) = \dfrac{t}{x(t)},\ x(1) = -2$

2.*

Solve the differential equation $x'(t) = a^2 - x^2(t)$ where $a = 2$ and
a) $x(0) = 1$ b) $x(0) = 3$

3. Autocatalytic reaction

First-order differential equations need not have solutions that are defined for all times.
a) Find the general solution of the equation $x'(t) = x(t)^2$.
b) Solve the initial value problems $x(0) = 1$ and $x(2) = -1$.
c) Discuss the domains over which each solution is defined.

Remark: The equation (where $x(t) \geq 0$ and $t \geq 0$) can be considered as the model of the autocatalytic reaction $2\,X + Y \to 3\,X$ where $x(t)$ denotes the concentration of the substance X and the concentration of Y is considered constant. Show that in this case the solution "blows up" in a finite time, that is, for all solution x, there exists a $T > 0$ such that $\lim\limits_{t \to T-} x(t) = +\infty$.

4. Radioactivity, exponential decay

Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. If $N(t)$ denotes the amount of radioactive material at time t then this process can be expressed by the differential equation

$$N'(t) = -\lambda\, N(t)$$

where $\lambda > 0$ is the exponential decay constant.
a) Find the general solution.
b) Show that there exists a number $T > 0$ (T is called the half-life) such that $N(t + T) = \dfrac{1}{2}\, N(t)$ for all $t > 0$. Express T by λ.
c) If the time is measured in hours, $\lambda = 10^{-3}$, the initial mass is $N(0) = 271$ kg then what will be the value of $N(t)$ after 1000 hours?
d) The the half-life of the radium is 1600 years. What percent of the initial quantity decays after 100 years?
e) A rock contains 100 mg of uranium and 14 mg of lead. The half-life of the uranium is $4.5 \cdot 10^9$ years and during the decay of 238 g of uranium, 206 g of lead is produced. How old is the rock?

5. Unlimited growth of bacteria

The unlimited growth of bacteria can be modelled with the equation $y'(t) = K\,y(t)$
where $K > 0$ and $y(t) \geq 0$ denotes the amount of bacteria at time t.
a) Find the general solution.
b) Find the solution if $y(0) = y_0$.

6. Heating problem

Newton’s law of cooling: The rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of its surroundings.
The process can be modelled by the equation

$$T'(t) = K(T_A - T(t))$$

where $K > 0$, $T_A > 0$ are constants. $T(t)$ denotes the temperature of the object at time t and T_A denotes the temperature of the surrounding medium.

a) When taken out of the oven, the temperature of a loaf decreases from 100 °C to 60 °C in 10 minutes. The temperature of the surrounding air is 20 °C. At what time will the temperature of the loaf reach 25 °C?

b) If the initial temperature of the loaf is 120 °C, the temperature of the air is $T_A = 30$ °C and $K = 0.0366$ then what will be the temperature of the loaf after 60 minutes?

7. Mixing problem

Suppose a brine containing 0.3 kg of salt per liter runs into a tank initially filled with 10 liters of water. The brine runs into the tank at the rate of 2 liter/min, the mixture is kept uniform by stirring, and the mixture flows out at the same rate. The equation giving the amount of salt in the tank at time t is $y'(t) = 0.6 - 0.2\,y(t)$.

a) Find the general solution. Show that the general solution converges to the constant solution.
b) How much salt is in the tank in 5 minutes?

8*. The logistic population model

The limited growth of bacteria can be modelled by the equation $y'(t) = K\,y(t)\,(M - y(t))$
where $K > 0$, $M > 0$ and M is the carrying capacity of the environment.
Find the general solutions $y(t) \geq 0$ and sketch the graph of the solutions.

9*. Parachutist

The equation $m\,v'(t) = m\,g - k\,v^2(t)$ describes the motion of a parachutist of mass m where $v(t)$ is the velocity at time t and $m\,v'(t)$ is mass times acceleration. By Newton’s second law, $m\,v'(t)$ equals the two forces, the attraction $m\,g$ by the earth and $-k\,v^2(t)$, the air resistance.

a) Find the general solution. Denoting the constant solutions by b, the equation can be transformed into the form

$$v'(t) = -\frac{k}{m}\left(v^2(t) - b^2\right). \quad \left(b = \sqrt{\frac{m\,g}{k}}\right)$$

b) Show that $v(t) \to b$ if $t \to +\infty$.

10. Fish population

A fish population in a lake can be modelled by the equation $x'(t) = K\,x(t) - H$ where $K > 0$ is the growth rate and $H > 0$ is the harvest quota. Let $x(0) = x_0 > 0$ be the initial population. Determine the values H of the harvest quota for which the population will not become extinct, that is, $x(t)$ is greater than a positive bound for all $t > 0$.

11. Outflow from a tank. Torricelli's law.

Torricelli's law states that water issues from a hole in the bottom of a tank with velocity $v(t) = 0.6\sqrt{2\,g\,h(t)}$ where $h(t)$ is the height of the water above the hole at time t and g is the acceleration of gravity. The outflow can be modelled by the equation

$$h'(t) = -A\sqrt{h(t)} \text{ where } A = \frac{r^2\,\pi}{R^2\,\pi} \cdot 0.6 \cdot \sqrt{2\,g}.$$

When will the tank be empty if $R = 0.9\,m$, $r = 3$ cm and $h(0) = 2.45\,m$?

Remark: If the flow is frictionless then $v(t) = \sqrt{2\,g\,h(t)}$. However, due to friction, the actual speed is lower. In the case of a circular opening, experience shows that $v(t) = 0.6\sqrt{2\,g\,h(t)}$.

12. Current in a closed RL circuit

The switch in an RL circuit is closed at time $t = 0$. The current I obeys the differential equation $L\,I'(t) + R\,I(t) = V$.

a) Find the solution if $I(0) = 0$.

b) How many seconds after the switch is closed will it take the current I to reach half of its steady state value?

13*. Snowballs

Two snowballs are thrown into the room. The radius of the bigger one is double the radius of the smaller one. The melting rate is proportional to the surface area. Find the radius of the bigger snowball at the time when the smaller one is completely melted.

14*. The logistic model with constant harvesting

Find the general solution of the logistic differential equation with constant harvesting

$$x'(t) = x(t)\,(1 - x(t)) - h$$

for all values of the parameter $h > 0$.