01 - Introduction

Basic concepts

Differential equation: an equation that contains an unknown function and its derivatives

Ordinary differential equation (ODE): the function has one variable
Partial differential equation (PDE): the function has two or more variables

nth order differential equation: the highest derivative is of order n

nth order implicit equation: $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$
nth order explicit equation: $y^{(n)} = f\left(x, y, y', y'', \ldots, y^{(n-1)}\right)$
nth order linear equation: $a_0(x)\, y(x) + a_1(x)\, y'(x) + a_2(x)\, y''(x) + \ldots + a_n(x)\, y^{(n)}(x) = f(x)$

Solution of a differential equation
In order to solve a differential equation, we want to find those differentiable one-variable real functions for which the equation holds, that is, substituting them into the equation, we get an identity.

General solution of an nth order differential equation: a family of functions with n parameters

The general solution in explicit form: $y = g(x, c_1, c_2, \ldots, c_n), \quad c_1, c_2, \ldots, c_n \in \mathbb{R}$
The general solution in implicit form: $G(x, y, c_1, c_2, \ldots, c_n) = 0, \quad c_1, c_2, \ldots, c_n \in \mathbb{R}$

Particular solution: one of the solutions

For example, setting n **initial conditions**, we can determine one particular solution:

$y(x_0) = y_{0,0}, \; y'(x_0) = y_{0,1}, \; \ldots, y^{(n-1)}(x_0) = y_{0,n-1}$ where $x_0, y_{0,0}, y_{0,1}, y_{0,n-1} \in \mathbb{R}$

Finding a particular solution is also possible with **boundary conditions**.

First-order ODE

Definition.

First order implicit differential equation:	(1) $F(x, y, y') = 0$ or: $F(x, y(x), y'(x)) = 0$
If y' can be expressed then the equation is explicit:	(2) $y' = f(x, y)$ or: $y'(x) = f(x, y(x))$
Initial value problem or Cauchy problem:	(3) $y' = f(x, y), \quad y(x_0) = y_0$

Definition.
a) We say that the differentiable function $\varphi : (a, b) \longrightarrow \mathbb{R}$, $(a < b)$ is a solution of (2) if

$\varphi'(x) = f(x, \varphi(x)), \quad \forall x \in (a, b)$

The graph of the solution or the integral curve is the graph of φ: $\{(x, y) : y = \varphi(x), x \in (a, b)\}$.

The general solution is the set of all solutions.
A particular solution is one of the solutions, for example the solution of the Cauchy problem.

b) We say that φ is a solution of the Cauchy problem (3) if there is an interval (a, b) (where $a < b$) such that

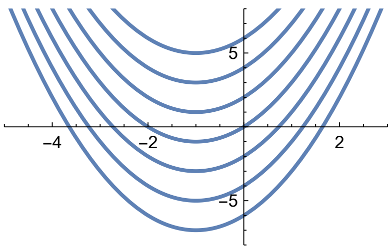
$\varphi'(x) = f(x, \varphi(x)), \quad \forall x \in (a, b)$ where $x_0 \in (a, b)$ and $\varphi(x_0) = y_0$

Example. $y' = 2x + 2$
a) Find the general solution of the differential equation.
b) Find the particular solution for which the initial condition $y(1) = 5$ holds.

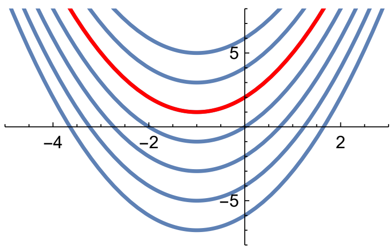
Solution. This is a first-order differential equation in explicit form.
a) The general solution: $y(x) = \int (2x + 2)\,dx = x^2 + 2x + C, \; C \in \mathbb{R}$
For each point of the plane, there is exactly one solution curve that passes through that point.

b) Here we solve an initial value problem where $x_0 = 1, \; y_0 = 5 \quad (y(x_0) = y_0)$
Substituting $x = 1$ and $y(1) = 5$ in the general solution:
$5 = 1 + 2 + C \implies C = 2$
The particular solution: $y(x) = x^2 + 2x + 2$
Here the solution of the initial value problem is unique.

a)



b)



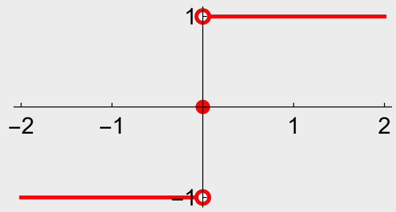
Basic questions

There are two basic questions relating to the theory of differential equations:
1) Existence: does an initial value problem have a solution?
2) Uniqueness: is the solution unique?

Existence and uniqueness of the solution

Exercise 1. Solve the following differential equation:

$$y'(x) = \text{sgn}(x) \quad \text{where} \quad \text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$



Solution. Since $f(x) = \text{sgn}(x)$ is not continuous then because of Darboux's theorem, the equation does not have a solution.
The theorem states that every function that is the derivative of another function
has the intermediate value property: the image of an interval is also an interval.

Darboux's theorem: Let I be a closed interval, $f : I \longrightarrow \mathbb{R}$ a real valued differentiable function.
Then f has the **intermediate value property**: if a and b are points in I with $a < b$, then
for every y between $f'(a)$ and $f'(b)$, there exists an x in (a, b) such that $f'(x) = y$.

Question: When does an initial value problem have a solution?

Cauchy-Peano theorem (or Peano existence theorem)

Theorem. Let I, J be intervals, $f : I \times J \longrightarrow \mathbb{R}$ and consider the initial value problem

$y'(x) = f(x, y(x)),\ \ y(x_0) = y_0$ (where $(x_0 \in I,\ y_0 \in J)$.

If ***f*** **is continuous** then for some value $\varepsilon > 0$ **there exists a solution** $\varphi(x)$ to the initial value problem on $[x_0 - \varepsilon, x_0 + \varepsilon]$:

$\varphi'(x) = f(x, \varphi(x)),\ \ \varphi(x_0) = y_0$ for all $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$.

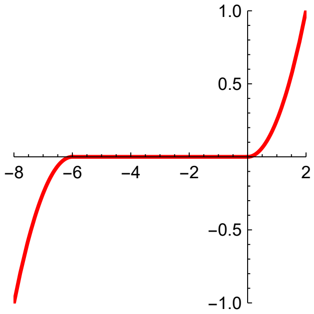
Remark: The solution need not be unique: the same initial value may give rise to many different solutions φ.

Exercise 2.

Let $y(x) = \begin{cases} \left(\frac{x}{2}\right)^2 & \text{if } x \geq 0 \\ 0 & \text{if } -6 \leq x \leq 0 \\ -\left(\frac{x}{2}+3\right)^2 & \text{if } x \leq -6 \end{cases}$

Show that y is continuously differentiable. Determine the functions y' and $\sqrt{|y|}$.

Solution. The graph of y:



y is differentiable on the open intervals $(-\infty, -6)$, $(-6, 0)$, $(0, \infty)$.

$$y'(x) = \begin{cases} 2 \cdot \frac{1}{2} \cdot \frac{x}{2} = \frac{x}{2} & \text{if } x > 0 \\ 0 & \text{if } -6 < x < 0 \\ -2 \cdot \frac{1}{2} \cdot \left(\frac{x}{2}+3\right) = -\left(\frac{x}{2}+3\right) & \text{if } x < -6 \end{cases}$$

y is also differentiable at $x = 0$ and $x = -6$.

By the definition of the derivative:

$$\lim_{h \to 0+0} \frac{y(0+h) - y(0)}{h} = \lim_{h \to 0+0} \frac{\left(\frac{h}{2}\right)^2 - 0}{h} = \lim_{h \to 0+0} \frac{h}{4} = 0 \text{ and}$$

$$\lim_{h \to 0-0} \frac{y(0+h) - y(0)}{h} = \lim_{h \to 0-0} \frac{0-0}{h} = 0 \implies y'(0) = 0 \text{ and similarly } y'(-6) = 0$$

$$\implies y \text{ is differentiable on } \mathbb{R} \text{ and } \; y'(x) = \begin{cases} \frac{x}{2} & \text{if } x \geq 0 \\ 0 & \text{if } -6 \leq x \leq 0 \\ -\left(\frac{x}{2}+3\right) & \text{if } x \leq -6 \end{cases}$$

Let us calculate $\sqrt{|y|}$:

$$\sqrt{|y(x)|} = \begin{cases} \sqrt{\left|\left(\frac{x}{2}\right)^2\right|} = \frac{x}{2} & \text{if } x \geq 0 \\ 0 & \text{if } -6 \leq x \leq 0 \\ \sqrt{\left|-\left(\frac{x}{2}+3\right)^2\right|} = \left|\frac{x}{2}+3\right| = -\left(\frac{x}{2}+3\right) & \text{if } x \leq -6 \end{cases}$$

$\implies$ y is a solution of the equation $y'(x) = \sqrt{|y(x)|}$.

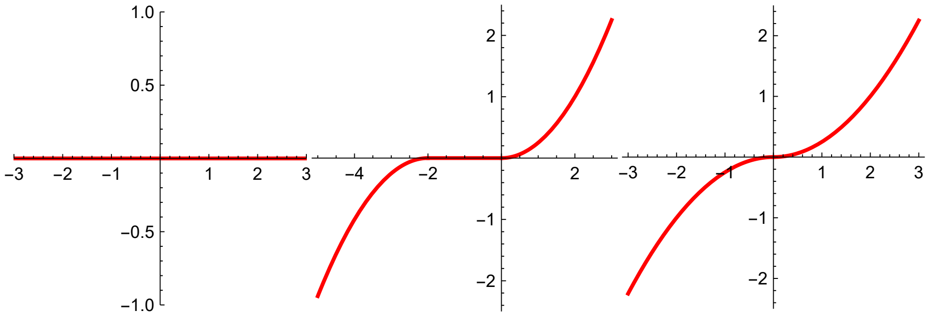
Exercise 3.
Solve the following initial value problem: $y'(x) = \sqrt{|\, y(x)\,|}\,,\quad y(0) = 0.$

Solution. The function y from the previous exercise is a solution since $y(0) = 0$.

Question: Are there any other solutions?

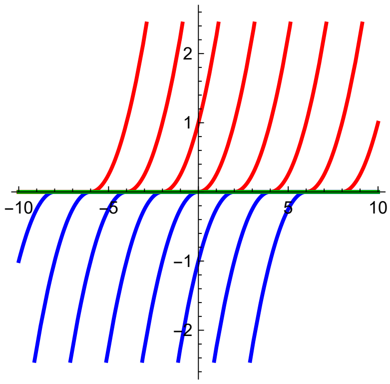
Other solutions for example: 1) $y \equiv 0$ (that is, $y(x) = 0,\ x \in \mathbb{R}$)

2) $y(x) = \begin{cases} \left(\frac{x}{2}\right)^2 & \text{if } x \geq 0 \\ 0 & \text{if } -2 \leq x \leq 0 \\ -\left(\frac{x}{2}+1\right)^2 & \text{if } x \leq -2 \end{cases}$ 3) $y(x) = \begin{cases} \left(\frac{x}{2}\right)^2 & \text{if } x \geq 0 \\ -\left(\frac{x}{2}\right)^2 & \text{if } x \leq 0 \end{cases}$



Question: How many solutions are there?

There are infinitely many solutions that can be constructed from the following functions:



Question: When is the solution unique?

Picard-Lindelöf theorem

Theorem. Let I, J be intervals, $f: I \times J \longrightarrow \mathbb{R}$ and consider the initial value problem

$y'(x) = f(x, y(x)),\ \ y(x_0) = y_0$

Suppose that ***f* is continuous in *x* and Lipschitz continuous in *y***, that is,
there exists a constant $L \geq 0$ such that for all $x \in I$ and $y_1, y_2 \in J$,

$| f(x, y_1) - f(x, y_2) | \leq L \, | y_1 - y_2 |.$

Then for some value $\varepsilon > 0$, **there exists a unique solution** $\varphi(x)$ to the initial value problem on $[x_0 - \varepsilon, x_0 + \varepsilon]$:

$\varphi'(x) = f(x, \varphi(x)),\ \ \varphi(x_0) = y_0$ for all $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$.

Remarks

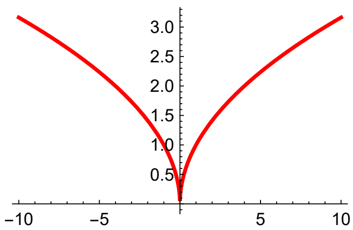
Remark 1. If f is differentiable with respect to y and the derivative is bounded then f is Lipschitz continuous is y.

Remark 2. Consider the initial value problem $y' = \sqrt{|y|},\ \ y(0) = 0$.

The function $f(x, y) = \sqrt{|y|}$ is not differentiable at $y = 0$ and is not Lipschitz continuous at $y = 0$.

The graph of $g(y) = \sqrt{|y|}$ has a vertical tangent line at $y = 0$.

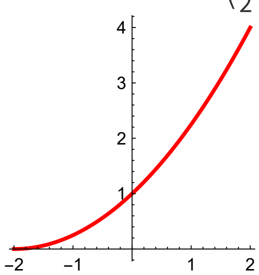
$\Longrightarrow$ the solution of the initial value problem is not unique



Remark 3. Consider the initial value problem $y' = \sqrt{|y|},\ \ y(0) = 1$.

The function $f(x, y) = \sqrt{|y|}$ is differentiable at $y = 1 \Longrightarrow$

the solution $y(x) = \left(\frac{x}{2} + 1\right)^2$ is unique if I and J are suitable neighbourhoods of $x_0 = 0$ and $y_0 = 1$.



Direction field or slope field

Definitions

Introduction. Suppose that $y = \varphi(x)$, $x \in (a, b)$ is a solution of the equation $y' = f(x, y)$. Then

$$\varphi'(x) = f(x, \varphi(x)), \quad \forall x \in (a, b).$$

If the graph of φ passes through the point (x_0, y_0) then $\varphi(x_0) = y_0$ and

$$\varphi'(x_0) = f(x_0, \varphi(x_0)) = f(x_0, y_0)$$

The meaning of $\varphi'(x_0)$ is the slope of the tangent line of the graph of φ that passes through the point (x_0, y_0), so $\tan \alpha = f(x_0, y_0)$.

Definitions. Consider the differential equation $y' = f(x, y)$.
1) A **lineal element** is a line segment with slope $f(x_0, y_0)$ passing through the point (x_0, y_0).
At each point (x_0, y_0), the lineal element is tangent to the solution curve passing through this point.

2) The **direction field** is the set of all lineal elements in the plane.
We can associate a direction field to every differential equation $y' = f(x, y)$.

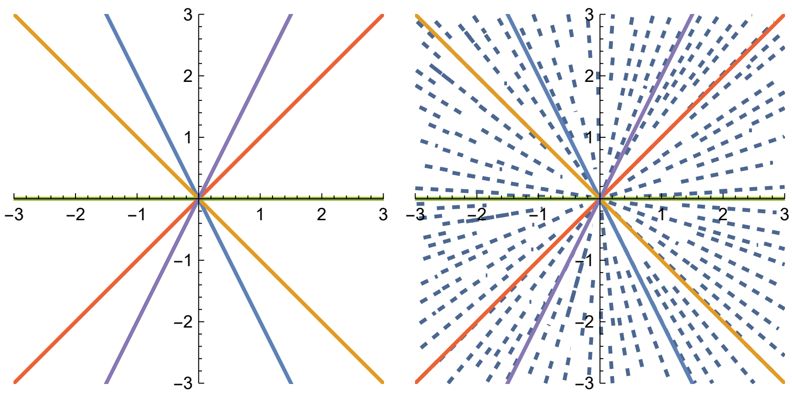
3) An **isocline** is a curve with equation $f(x, y) = K$, $K \in \mathbb{R}$.
That is, all the solution intersecting that curve have the same slope K.

Examples

Example 1. $y' = \frac{y}{x}$

Isoclines: $y' = K$ $(K \in \mathbb{R}) \Longrightarrow \frac{y}{x} = K, y = K x$ (straight lines)

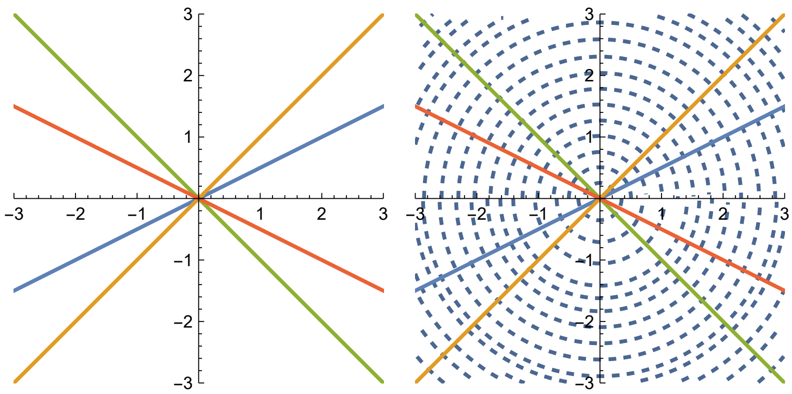
Isoclines with $K = -2, -1, 0, 1, 2$ and the direction field:



Example 2. $y' = -\dfrac{x}{y}$

Isoclines: $y' = K \;\; (K \in \mathbb{R}) \implies -\dfrac{x}{y} = K,\; y = -\dfrac{1}{K}\, x$ (straight lines)

The isoclines with $K = -2, -1, 1, 2$ and the direction field:



Remark: The differential equation can be obtained from the equation of the circle:

$x^2 + y^2 = C$ (or: $x^2 + y^2(x) = C$) $\implies y(x) = \pm \sqrt{C - x^2}$

Differentiating both sides with respect to x: $2x + 2\, y(x)\, y'(x) = 0 \implies y'(x) = -\dfrac{x}{y(x)}$

Verification:

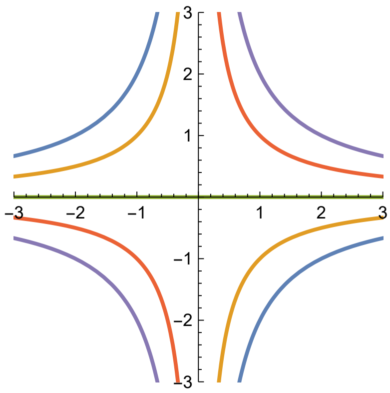
$$y(x) = \sqrt{C - x^2} \implies y'(x) = \left(\sqrt{C - x^2}\right)' = \frac{1}{2}\,(C - x^2)^{-1/2} \cdot (-2x) = -\frac{x}{\sqrt{C - x^2}} = -\frac{x}{y(x)}$$

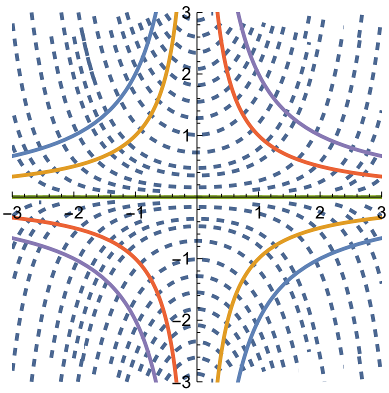
$$y(x) = -\sqrt{C - x^2} \implies y'(x) = \left(-\sqrt{C - x^2}\right)' = -\frac{1}{2}\,(C - x^2)^{-1/2} \cdot (-2x) = \frac{x}{\sqrt{C - x^2}} = -\frac{x}{-\sqrt{C - x^2}} = -\frac{x}{y(x)}$$

Example 3. $y' = x\,y$

Isoclines: $y' = K \;\; (K \in \mathbb{R}) \implies x\,y = K,\; y = K \cdot \dfrac{1}{x}$ (hyperbolas)

Isoclines with $K = -2, -1, 0, 1, 2$ and the direction field:





Remark. Recall that f has a local extremum at x_0 if $f'(x_0) = 0$ and $f''(x_0) \neq 0$ (local minimum if $f''(x_0) > 0$ and local maximum if $f''(x_0) < 0$).
f has an inflection point at x_0 if $f''(x_0) = 0$ and $f'''(x_0) \neq 0$.

Example 4*. Consider the equation $y' = x - y^2$.

1) Find the equation of the isoclines. At what points does the solution have an extremum and what is the type of the extremum?
2) Consider the solution passing through the point $(1, -2)$. Does this solution have an inflection point at $(1, -2)$?

Solution.

1) The equation of the isoclines: $x - y^2 = K$ (lying parabolas)

Necessary condition for the local extremum: $y' = 0$

$\Longrightarrow$ the solutions can have a local extremum at the points of the isocline with $K = 0$:

$x - y^2 = 0 \Longrightarrow x = y^2$ (the solution curves intersect this isocline horizontally)

For the sufficient condition we investigate the value of y''.

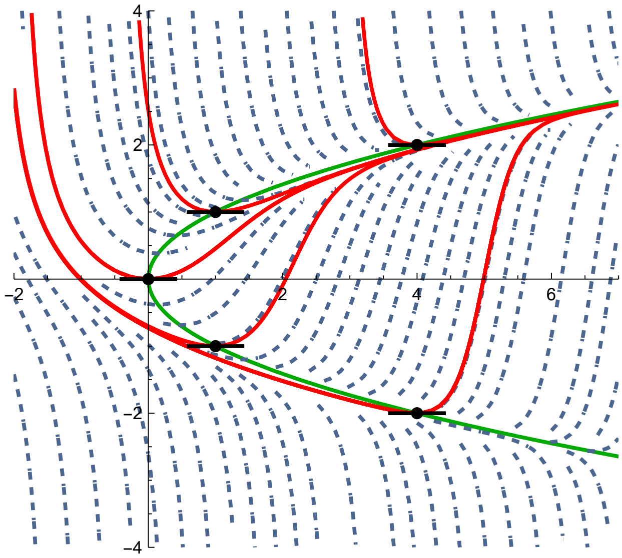
Differentiating both sides of the equation $y'(x) = x - y^2(x)$ with respect to x:

$y'' = 1 - 2\,y\,y'$

If $x = y^2$ then $y' = 0 \Longrightarrow y'' = 1 > 0$

Thus, at the points of the parabola $x = y^2$ we have that $y' = 0$ and $y'' > 0$

$\Longrightarrow$ the solution curves have a local minimum at the points of the parabola.



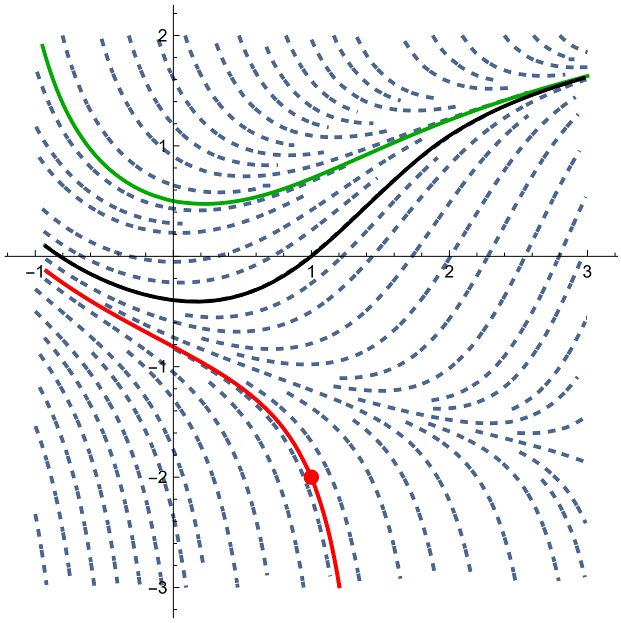
2) $y(1) = -2$ and $y' = x - y^2 \implies y'(1) = 1 - (-2)^2 = -3$

$y'' = 1 - 2\,y\,y' \implies y''(1) = 1 - 2 \cdot (-2) \cdot (-3) = -11 \neq 0$

Necessary condition for the existence of an inflection point doesn't hold

$\implies$ the solution doesn't have an inflection point at (1, −2).

Some solution curves: $y(0) = 0.5$ (green), $y(1) = 0$ (black), $y(1) = -2$ (red)



Example 5. $y' = x^3 + y^3 - 9$

Consider the solution that passes through the point (2, 1). What local properties does this solution have at (2, 1)?

Solution.

The solution passes through (2, 1) $\implies y(2) = 1 \quad (x_0 = 2,\ y_0 = 1)$

$y'(x) = x^3 + y^3(x) - 9 \implies y'(2) = 2^3 + 1^3 - 9 = 0$

$\implies$ the solution can have a local extremum at $x_0 = 2$

Differentiating both sides with respect to x:

$y''(x) = 3x^2 + 3y^2(x)\,y'(x) \implies y''(2) = 3 \cdot 2^2 + 3 \cdot 1^2 \cdot 0 = 12 > 0$

Since $y'(2) = 0$ and $y''(2) > 0 \implies y(2) = 1$ is a local minimum