01 - Introduction, exercises

Integration methods

1. Some basic integrals

1. $\int x^\alpha \,\mathrm{dx} = \frac{x^{\alpha+1}}{\alpha+1} + c \;\; (\alpha \neq -1)$
2. $\int \frac{1}{x} \,\mathrm{dx} = \ln|x| + c$
3. $\int e^x \,\mathrm{dx} = e^x + c$
4. $\int \cos \mathrm{dx} = \sin x + c$
5. $\int \sin x \,\mathrm{dx} = -\cos x + c$

2. Linear substitution: $\int f(a\,x+b)\,\mathrm{dx} = \frac{F(a\,x+b)}{a} + c$, where $F' = f$ and $a \neq 0$

Examples:

1. $\int \frac{1}{3\,x+2}\,\mathrm{dx} = \frac{\ln|3\,x+2|}{3} + c$, since $\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\ln(3\,x+2)}{3} + c\right) = \frac{1}{3}\,\frac{1}{3\,x+2}\cdot 3 = \frac{1}{3\,x+2}$
2. $\int \frac{1}{4-x}\,\mathrm{dx} = \frac{\ln|4-x|}{-1} + c = -\ln|4-x| + c$
3. $\int e^{2\,x}\,\mathrm{dx} = \frac{e^{2\,x}}{2} + c$
4. $\int e^{-x}\,\mathrm{dx} = \frac{e^{-x}}{-1} + c = -e^{-x} + c$
5. $\int \sin(2\,x)\,\mathrm{dx} = -\frac{\cos(2\,x)}{2} + c$

3. Polynomial in the denominator

Examples:

1. $\int \frac{1}{a\,x+b}\,\mathrm{dx} = \frac{\ln|a\,x+b|}{a} + c \;\; (a \neq 0)$

2. $I = \int \frac{1}{(x-a)\,(x-b)}\,\mathrm{dx} = ?$

Suppose that $a \neq b \implies$ the denominator has two different real roots.
Partial fraction decomposition:

$$\frac{1}{(x-a)\,(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} = \frac{A(x-b)+B(x-a)}{(x-a)\,(x-b)}$$

$\implies 1 = A(x-b) + B(x-a)$
We want to determine A and B such that the equation holds for all $x \in \mathbb{R}$.

1st method (comparing the coefficients):
$1 = (A+B)\,x + (-A\,b - B\,a) \quad$ (for all $x \in \mathbb{R}$)

$\Longrightarrow \quad A + B = 0 \Longrightarrow B = -A \Longrightarrow -A\,b + A\,a = A(a-b) = 1$

$-A\,b - B\,a = 1$

$\Longrightarrow A = \frac{1}{a-b}, \; B = -\frac{1}{a-b}$

2nd method (substitution method):

$1 = A(x-b) + B(x-a) \quad$ (for all $x \in \mathbb{R}$)

$\Longrightarrow x = b: \; 1 = A \cdot 0 + B(b-a) \Longrightarrow B = \frac{1}{b-a} = -\frac{1}{a-b}$

$x = a: \; 1 = A(a-b) + B \cdot 0 \Longrightarrow A = \frac{1}{a-b}$

The integral: $I = \int \frac{1}{(x-a)(x-b)}\,\mathrm{dx} =$

$= \int \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)\mathrm{dx} = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + c = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + c$

$= \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + c = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + c$

3. Special case ($b = -a$):

$\int \frac{1}{x^2 - a^2}\,\mathrm{dx} = \int \frac{1}{(x-a)(x+a)}\,\mathrm{dx} = \frac{1}{2a}(\ln|x-a| - \ln|x+a|) + c = \frac{1}{2a}\ln\left|\frac{x-a}{x-b}\right| + c$

4. Integration by parts: $\int f' \cdot g = f \cdot g - \int f \cdot g'$

Example: $\int x\,e^x\,\mathrm{dx} = e^x \cdot x - \int e^x \cdot 1\,\mathrm{dx} = x\,e^x - e^x + c$

$f'(x) = e^x \Longrightarrow f(x) = e^x$

$g(x) = x \Longrightarrow g'(x) = 1$

Verification

5. Verify that the given function is a solution.

a) $y'(x) + y(x) = x^2 - 2, \qquad y(x) = c\,e^{-x} + x^2 - 2x$

b) $y''(x) + y(x) = 0, \qquad y(x) = a\cos x + b\sin x$

c) $y'''(x) = e^x, \qquad y(x) = e^x + a\,x^2 + b\,x + c$

d) $y''(x) + 2\,y'(x) + 2\,y(x) = 0, \qquad y(x) = e^{-x}(a\cos x + b\sin x)$

e) $y'(x) + 2\,y(x) = e^x, \qquad y(x) = -2\,e^{-2x} + \frac{1}{3}e^x$