Calculus 2, Midterm Test 2

25th May, 2023

1. (16 points) Let $f(x, y) = \begin{cases} \dfrac{(y+1)\,x^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

a) Calculate the partial derivatives of f where they exist. (At the origin use the definition.)

c) Where is f differentiable?

2. (18 points) Let $f(x, y) = \begin{cases} y - \dfrac{y^2}{x^2+y^2} - 2\,x \sin\left(\dfrac{\pi}{1+y^2}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

a) Calculate the partial derivatives of f if $(x, y) \neq (0, 0)$.

b) Show that f is differentiable at the point $P(1, 1)$.

c) Find the equation of the tangent plane of f at the point $P(1, 1)$.

d) Find the directional derivative of f at $P(1, 1)$ in the direction $\mathbf{v} = (1, 2)$.

3. (16 points) Determine the Taylor polynomial of order 2 of the function $f(x, y) = x^2 + y^2 \arctan x$ at the point $P(1, 1)$.

4. (16 points) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = \left(y\, e^{x+2y},\ 3\sqrt{x^2+5\,y^2}\right)$. Show that in a small neighbourhood of $f(2, -1) = (-1, 9)$ the inverse function f^{-1} exists and determine the derivative $\left(f^{-1}\right)'(-1, 9)$.

5. (18 points) Determine the local minima and maxima of the function $f(x, y) = 2\,x^2\,y + 2\,x\,y - 3\,y^2$

6. (16 points) Determine the maximum and minimum of the function $f(x, y, z) = x^2 + y^2 + z^2$ under the constraint $z^2 = x^2\,y + 4$

7.* (10 points - BONUS)

Let $f(x, y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Using the definition, show that f is not differentiable at the origin.

Solutions

1. (16 points) Let $f(x, y) = \begin{cases} \dfrac{(y+1)\,x^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

a) Calculate the partial derivatives of f where they exist. (At the origin use the definition.)

c) Where is f differentiable?

Solution.

a) If $(x, y) \neq (0, 0)$ then

$$f'_x(x, y) = \frac{2\,x\,(y+1)\left(x^2+y^2\right) - (y+1)\cdot x^2\cdot 2\,x}{\left(x^2+y^2\right)^2}, \quad f'_y(x, y) = \frac{x^2\left(x^2+y^2\right) - x^2(y+1)\cdot 2\,y}{\left(x^2+y^2\right)^2} \quad \textbf{(6p)}$$

If $(x, y) = (0, 0)$ then

$$f'_x(0, 0) = \lim_{h\to 0}\frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0}\frac{\frac{(0+1)\,h^2}{h^2+0} - 0}{h} = \lim_{h\to 0}\frac{1}{h}, \text{ it doesn't exist.}$$

$$f'_y(0, 0) = \lim_{h\to 0}\frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0}\frac{\frac{(h+1)\cdot 0}{0+h^2} - 0}{h} = 0. \quad \textbf{(6p)}$$

b) Outside of the origin the partial derivatives are continuous, so f is differentiable on this open set. At the origin f is not differentiable, since $f'_x(0, 0)$ doesn't exist. **(4p)**

2. (18 points) Let $f(x, y) = \begin{cases} y - \dfrac{y^2}{x^2+y^2} - 2\,x\sin\left(\dfrac{\pi}{1+y^2}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

a) Calculate the partial derivatives of f if $(x, y) \neq (0, 0)$.

b) Show that f is differentiable at the point $P(1, 1)$.

c) Find the equation of the tangent plane of f at the point $P(1, 1)$.

d) Find the directional derivative of f at $P(1, 1)$ in the direction $\boldsymbol{v} = (1, 2)$.

Solution. a) $f'_x(x, y) = \dfrac{2\,x\,y^2}{x^2+y^2} - 2\sin\left(\dfrac{\pi}{1+y^2}\right)$

$$f'_y(x, y) = 1 - \frac{2\,x^2\,y}{\left(x^2+y^2\right)^2} + \frac{4\,\pi\,x\,y}{\left(1+y^2\right)^2}\cos\left(\frac{\pi}{1+y^2}\right) \quad \textbf{(6p)}$$

b) The partial derivatives are continuous at $P(1, 1)$, so f is differentiable at this point. **(2p)**

c) The equation of the tangent plane is $z = f(a, b) + f'_x(a, b)\,(x-a) + f'_y(a, b)\,(y-b)$.

Here $(a, b) = (1, 1) \Longrightarrow f(1, 1) = -\frac{3}{2}$, $f'_x(1, 1) = -\frac{3}{2}$, $f'_y(1, 1) = \frac{1}{2}$

$\Longrightarrow$ the equation of the tangent plane is $z = -\frac{3}{2} - \frac{3}{2}(x-1) + \frac{1}{2}(y-1)$. **(5p)**

d) Since $\|\,\boldsymbol{v}\,\| = \sqrt{1^2+2^2} = \sqrt{5}$ then the unit vector parallel to $\boldsymbol{v}$ is $\boldsymbol{e} = \dfrac{1}{\|\,\boldsymbol{v}\,\|}\cdot\boldsymbol{v} = \left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$.

The gradient vector of f at $P(1, 1)$ is $\operatorname{grad} f(1, 1) = \left(-\frac{3}{2}, \frac{1}{2}\right)$.

The directional derivative of f at $P(1, 1)$ in the direction $\boldsymbol{v} = (1, 2)$ is

$$\boldsymbol{e} \cdot \operatorname{grad} f(1, 1) = < \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(-\frac{3}{2}, \frac{1}{2}\right) > = \frac{-3}{2\sqrt{5}} + \frac{2}{2\sqrt{5}} = -\frac{1}{2\sqrt{5}} \quad \textbf{(5p)}$$

3. (16 points) Determine the Taylor polynomial of order 2 of the function $f(x, y) = x^2 + y^2 \arctan x$ at the point $P(1, 1)$.

Solution. Calculate the first-order and second order partial derivatives of f and evaluate them at the given point:

$$f(x, y) = x^2 + y^2 \arctan x \quad \Longrightarrow f(1, 1) = 1 + \frac{\pi}{4}$$

$$f'_x(x, y) = 2x + \frac{y^2}{1 + x^2} \quad \Longrightarrow f'_x(1, 1) = \frac{5}{2}$$

$$f'_y(x, y) = 2y \arctan x \quad \Longrightarrow f'_y(1, 1) = \frac{\pi}{2}$$

$$f''_{xx}(x, y) = 2 - \frac{2xy^2}{(1 + x^2)^2} \quad \Longrightarrow f''_{xx}(1, 1) = \frac{3}{2}$$

$$f''_{xy}(x, y) = \frac{2y}{1 + x^2} \quad \Longrightarrow f''_{xy}(1, 1) = 1$$

$$f''_{yx}(x, y) = \frac{2y}{1 + x^2} \quad \Longrightarrow f''_{yx}(1, 1) = 1$$

$$f''_{yy}(x, y) = 2 \arctan x \quad \Longrightarrow f''_{yy}(1, 1) = \frac{\pi}{2} \quad \textbf{(8p)}$$

The Taylor polynomial of order 2 at a point (a, b) is

$$T_2(x, y) = f(a, b) + f'_x(a, b)(x - a) + f'_y(a, b)(y - b) +$$

$$+ \frac{1}{2}\left(f''_{xx}(a, b)(x - a)^2 + 2 f''_{xy}(a, b)(x - a)(y - b) + f''_{yy}(a, b)(y - b)^2\right)$$

Substituting $(a, b) = (1, 1)$:

$$f(x, y) \approx T_2(x, y) = \left(1 + \frac{\pi}{4}\right) + \frac{5}{2}(x - 1) + \frac{\pi}{2}(y - 1) + \frac{1}{2}\left(\frac{3}{2}(x - 1)^2 + 2(x - a)(y - b) + \frac{\pi}{2}(y - 1)^2\right) \quad \textbf{(8p)}$$

4. (16 points) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = \left(y\, e^{x+2y}, 3\sqrt{x^2 + 5y^2}\right)$. Show that in a small neighbourhood of $f(2, -1) = (-1, 9)$ the inverse function f^{-1} exists and determine the derivative $\left(f^{-1}\right)'(-1, 9)$.

Solution. Calculating the Jacobian matrix of f:

$$f'(x, y) = \begin{pmatrix} \frac{\partial f_1(x, y)}{\partial x} & \frac{\partial f_1(x, y)}{\partial y} \\ \frac{\partial f_2(x, y)}{\partial x} & \frac{\partial f_2(x, y)}{\partial y} \end{pmatrix} = \begin{pmatrix} y\, e^{x+2y} & e^{x+2y} + 2y\, e^{x+2y} \\ \frac{3x}{\sqrt{x^2 + 5y^2}} & \frac{15y}{\sqrt{x^2 + 5y^2}} \end{pmatrix} \quad \textbf{(6p)}$$

Substituting $(x, y) = (2, -1)$ into the Jacobian:

$f'(2, -1) = \begin{pmatrix} -1 & -1 \\ 2 & -5 \end{pmatrix}$ **(2p)**

Calculating the determinant of the Jacobian:
$\det f'(2, -1) = \det\begin{pmatrix} -1 & -1 \\ 2 & -5 \end{pmatrix} = 5 + 2 = 7 \neq 0$

Since the above determinant is not zero then by the inverse function theorem,
the inverse function f^{-1} exists in a small neighbourhood if the point $f(2, -1) = (-1, 9)$. **(4p)**

The derivative of f^{-1} at $(-1, 9)$ is the inverse of $f'(2, -1)$.
$(f^{-1})'(-1, 9) = (f'(2, -1))^{-1} = \frac{1}{7}\begin{pmatrix} -5 & 1 \\ -2 & -1 \end{pmatrix}$. **(4p)**

5. (18 points) Determine the local minima and maxima of the function $f(x, y) = 2x^2 y + 2xy - 3y^2$

Solution. The first-order partial derivatives of f are:
(1) $f'_x(x, y) = 4xy + 2y = 2y(2x + 1) = 0 \quad \Longrightarrow \quad y = 0$ or $x = -\frac{1}{2}$
(2) $f'_y(x, y) = 2x^2 + 2x - 6y = 0$
Case 1. If $y = 0$ then $2x^2 + 2x = 2x(x + 1) = 0 \quad \Longrightarrow \quad x = 0$ or $x = -1$
Case 2. If $x = -\frac{1}{2}$ then $y = \frac{2x^2 + 2x}{6} = -\frac{1}{12}$
The stationary points are: $P_1(0, 0)$, $P_2(-1, 0)$, $P_3\left(-\frac{1}{2}, -\frac{1}{12}\right)$. **(8p)**

The Hesse-matrix of f is $H(x, y) = \begin{pmatrix} f''_{xx}(x, y) & f''_{xy}(x, y) \\ f''_{yx}(x, y) & f''_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} 4y & 4x + 2 \\ 4x + 2 & -6 \end{pmatrix}$ **(3p)**
Evaluating the Hesse-matrix at the given points:
At $P_1(0, 0)$: $H(0, 0) = \begin{pmatrix} 0 & 2 \\ 2 & -6 \end{pmatrix}$. Since $\det H(P_1) = -4 < 0$ then P_1 is a saddle point. **(2p)**
At $P_2(-1, 0)$: $H(-1, 0) = \begin{pmatrix} 0 & -2 \\ -2 & -6 \end{pmatrix}$. Since $\det H(P_2) = -4 < 0$ then P_1 is a saddle point. **(2p)**
At $P_3\left(-\frac{1}{2}, -\frac{1}{12}\right)$: $H\left(-\frac{1}{2}, -\frac{1}{12}\right) = \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & -6 \end{pmatrix}$. Since $\det H(P_3) = 2 > 0$ and $f''_{xx}\left(-\frac{1}{2}, -\frac{1}{12}\right) = -\frac{1}{3} < 0$
then f has a local maximum at P_3. **(3p)**

6. (16 points) Determine the maximum and minimum of the function $f(x, y, z) = x^2 + y^2 + z^2$ under the constraint $z^2 = x^2 y + 4$

Solution. Substituting $z^2 = x^2 y + 4$ into $f(x, y, z)$, we obtain the function
$g(x, y) = x^2 + y^2 + x^2 y + 4$.

The first-order partial derivatives of g are:
(1) $g'_x(x, y) = 2x + 2xy = 2x(1 + y) = 0 \quad \Longrightarrow \quad x = 0$ or $y = -1$
(2) $g'_y(x, y) = 2y + x^2 = 0$

If $x = 0$ then $y = 0$.

If $y = -1$ then $x^2 = 2 \Longrightarrow x = \pm\sqrt{2}$.

The stationary points are: $P_1(0, 0)$ and $P_1\left(\sqrt{2}, -1\right)$, $P_3\left(-\sqrt{2}, -1\right)$. **(6p)**

The Hesse-matrix of g is $H(x, y) = \begin{pmatrix} g''_{xx}(x, y) & g''_{xy}(x, y) \\ g''_{yx}(x, y) & g''_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} 2+2y & 2x \\ 2x & 2 \end{pmatrix}$ **(3p)**

Evaluating the Hesse-matrix at the given points:

At $P_1(0, 0)$: $H(P_1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Since $\det H(P_1) = 4 > 0$ and $f''_{xx}(P_1) = 2 > 0$ then

f has a local minimum at $P_1(0, 0)$ and $z = \pm 2$. **(3p)**

At P_2 and P_3: $H\left(\pm\sqrt{2}, -1\right) = \begin{pmatrix} 0 & \pm 2\sqrt{2} \\ \pm 2\sqrt{2} & 2 \end{pmatrix}$. Since $\det H\left(\pm\sqrt{2}, -1\right) = -8 < 0$ then

P_2 and P_3 are saddle points. **(3p)**

Therefore, f has a local minimum at the points $(0, 0, 2)$ and $(0, 0, -2)$ and the value of the minimum is 4. **(1p)**

7.* (10 points - BONUS)

Let $f(x, y) = \begin{cases} \dfrac{x^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Using the definition, show that f is not differentiable at the origin.

Solution. The partial derivatives at the origin are:

$$f'_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{h^3}{h^2 + 0} - 0\right) = 1$$

$$f'_y(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{1}{h}(0 - 0) = 0$$

If grad $f(0, 0)$ exists then its can only be $(1, 0)$.

Calculate the following limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x, y) - f(0, 0) - \langle (1, 0), (x, y) \rangle}{\|(x, y)\|} =$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^3}{x^2 + y^2} - 0 - x}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{-x y^2}{x^2 + y^2} \cdot \frac{1}{\sqrt{x^2 + y^2}} =$$

$$= \lim_{r\to 0} \frac{-r^3 \cos\varphi \sin^2\varphi}{r^3} \cdot \frac{1}{\sqrt{x^2 + y^2}} = -\cos\varphi \sin^2\varphi,$$ this depends on φ.

Since $\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x, y) - f(0, 0) - \langle (1, 0), (x, y) \rangle}{\|(x, y)\|} \neq 0$ then grad $f(0, 0)$ doesn't exist.