Calculus 1, Midterm Test 1

20th April, 2023

1. (10+5 points) **a)** Calculate the value of the following integral: $\int_0^1 x \ln x \, \mathrm{d}x$

b) Decide whether the following integral converges or diverges: $\int_1^\infty \frac{x^2 + \sqrt{x}\,\sin(x)}{x^5+2}\,\mathrm{d}x$

2. (5+5+5 points) Let $f_n(x) = \frac{n\,x^2}{1+n\,x^2}$ for $x \in \mathbb{R}$.

a) Determine the pointwise limit of $f_n(x)$ on $\mathbb{R}$.

b) Decide whether the convergence is uniform on $\mathbb{R}$.

c) Decide whether the convergence is uniform on $[1, \infty)$.

3. (5+5+5 points) Let $f_n(x) = \frac{\arctan(n\,x)}{2^n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

a) Show that the series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent for all $x \in \mathbb{R}$.

b) Let $S(x) = \sum_{n=0}^{\infty} f_n(x)$ for all $x \in \mathbb{R}$. For what values of $x \in \mathbb{R}$ is S differentiable? Calculate $S'(x)$.

c) Show that $\sum_{n=1}^{\infty} n\,x^n = \frac{x}{(1-x)^2}$ if $|x| < 1$. Using this, calculate the value of $S'(0)$.

4. (6+6 points) Find the Taylor series of the following functions at $x_0 = 3$ and determine the radius of convergence.

a) $f(x) = \frac{1}{x+7}$ **b)** $g(x) = e^{2x+1}$

5. (6+6+6 points) Let $f(x) = \frac{1}{\sqrt{16+x^4}}$.

a) Find the Taylor series of f at $x_0 = 0$ and determine the radius of convergence.

b) Calculate $f^{(15)}(0)$ and $f^{(16)}(0)$.

c) Using part a), determine the approximate value of the integral $f(x) = \int_0^1 f(x)\,\mathrm{d}x$ such that f is approximated by its Taylor polynomial of order 4. Give an estimation for the error.

6. (4+6+5 points) Let $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < 1,\ |y| < x^2\}$.

a) Sketch the set A.

b) Find the set of interior points and the set of boundary points of A.

c) Find the set of limit points and the closure of A.

7. (8+7 points) Calculate the following limits if they exist.

a) $\lim_{(x,y)\to(0,0)} \frac{x^2\,y^2}{x^2+y^2}$ **b)** $\lim_{(x,y)\to(0,0)} \frac{\sin(x\,y)}{x^2+y^2}$

8.* (10 points - BONUS)

Prove that $\sum_{n=1}^{\infty}\left(1-\cos\frac{x}{n}\right)$ converges uniformly and is differentiable on any bounded interval $(a, b) \subset \mathbb{R}$. (Help: use Lagrange's mean value theorem to get a bound for $\left|1-\cos\frac{x}{n}\right|$.)

Solutions

1. (10+5 points)

a) Calculate the value of the following integral: $\int_0^1 x \ln x \, dx$

b) Decide whether the following integral converges or diverges: $\int_1^\infty \frac{x^2 + \sqrt{x}\,\sin(x)}{x^5 + 2} \, dx$

Solution.

a) With the integration by party method: $\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$

Using this, the improper integral is

$$\int_0^1 x \ln x \, dx = \lim_{\varepsilon \to 0+} \int_\varepsilon^1 x \ln x \, dx = \lim_{\varepsilon \to 0+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_\varepsilon^1 = \lim_{\varepsilon \to 0+} \left(\left(\frac{1}{2} \ln 1 - \frac{1}{4} \right) - \left(\frac{\varepsilon^2}{2} \ln \varepsilon - \frac{\varepsilon^2}{4} \right) \right) =$$

$$= \left(0 - \frac{1}{4} \right) - (0 - 0) = -\frac{1}{4}.$$

Here we use that $\lim_{\varepsilon \to 0+} \frac{\varepsilon^2}{2} \ln \varepsilon = \lim_{\varepsilon \to 0+} \frac{\ln \varepsilon}{2 \varepsilon^{-2}} \overset{"\frac{-\infty}{\infty}", L'H}{=} \lim_{\varepsilon \to 0+} \frac{\frac{1}{\varepsilon}}{-4 \varepsilon^{-3}} = \lim_{\varepsilon \to 0+} \frac{\varepsilon^2}{-4} = 0.$

b) Let $f(x) = \frac{x^2 + \sqrt{x}\,\sin(x)}{x^5 + 2}$. If $x \geq 1$ then

$$|f(x)| \leq \frac{x^2 + \sqrt{x}\,|\sin(x)|}{x^5 + 2} \leq \frac{x^2 + \sqrt{x} \cdot 1}{x^5 + 2} \leq \frac{x^2 + x^2}{x^5 + 0} = \frac{2}{x^3}.$$

Since $\int_1^\infty \frac{2}{x^3} \, dx$ is convergent, then $\int_1^\infty \frac{x^2 + \sqrt{x}\,\sin(x)}{x^5 + 2} \, dx$ is also convergent.

2. (5+5+5 points) Let $f_n(x) = \frac{n x^2}{1 + n x^2}$ for $x \in \mathbb{R}$.

a) Determine the pointwise limit of $f_n(x)$ on $\mathbb{R}$.

b) Decide whether the convergence is uniform on $\mathbb{R}$.

c) Decide whether the convergence is uniform on $[1, \infty)$.

Solution.

a) If $x = 0$ then $f_n(0) = 0$. If $x \neq 0$ then $\lim_{n \to \infty} f_n(x) = \frac{n}{n} \cdot \frac{x^2}{\frac{1}{n} + x^2} = \frac{x^2}{0 + x^2} = 1.$

The limit function is $f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$

b) The convergence of (f_n) to f on $\mathbb{R}$ is not uniform, since f has a discontinuity at $x = 0$, but f_n is continuous on $\mathbb{R}$ for all $n \in \mathbb{N}$.

Or: We can use that if there exists a sequence $(x_n) \subset H$ such that $\lim_{n \to \infty} |r_n(x_n)| = \lim_{n \to \infty} |f_n(x_n) - f(x_n)| \neq 0$, then (f_n) does not converge uniformly to f on H.

Here let $x_n = \frac{1}{n}$, then $\lim\limits_{n\to\infty} |\, r_n(x_n)\, | = \lim\limits_{n\to\infty} |\, f_n(x_n) - f(x_n)\, | = \lim\limits_{n\to\infty} \left| \frac{n\,x_n^2}{1+n\,x_n^2} - 1 \right| =$

$$= \lim_{n\to\infty} \left| \frac{n\,x_n^2 - \left(1+n\,x_n^2\right)}{1+n\,x_n^2} \right| = \lim_{n\to\infty} \frac{1}{1+n\,x_n^2} = \lim_{n\to\infty} \frac{1}{1+n\cdot\frac{1}{n^2}} = \frac{1}{1+0} = 1 \neq 0$$

$\Longrightarrow$ (f_n) does not converge uniformly to f on $\mathbb{R}$.

c) To show uniform convergence we use that if there exists a sequence (c_n) such that $|\, r_n(x)\, | = |\, f_n(x) - f(x)\, | \leq c_n$ on H for $n > n_0$ and $\lim\limits_{n\to\infty} c_n = 0$,

then (f_n) converges uniformly to f on H.

If $x \geq 1$, then $|\, r_n(x)\, | = \left| \frac{n\,x^2}{1+n\,x^2} - 1 \right| = \left| \frac{-1}{1+n\,x^2} \right| = \frac{1}{1+n\,x^2} \leq \frac{1}{1+n} \longrightarrow 0$,

so (f_n) converges uniformly to f on $[1, \infty)$.

3. (5+5+5 points)

Let $f_n(x) = \frac{\arctan(n\,x)}{2^n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

a) Show that the series $\sum\limits_{n=0}^{\infty} f_n(x)$ is uniformly convergent for all $x \in \mathbb{R}$.

b) Let $S(x) = \sum\limits_{n=0}^{\infty} f_n(x)$ for all $x \in \mathbb{R}$. For what values of $x \in \mathbb{R}$ is S differentiable? Calculate $S'(x)$.

c) Show that $\sum\limits_{n=1}^{\infty} n\,x^n = \frac{x}{(1-x)^2}$ if $|\, x\, | < 1$. Using this, calculate the value of $S'(0)$.

Solution.

a) Since $|\, f_n(x)\, | = \left| \frac{\arctan(n\,x)}{2^n} \right| < \frac{1}{2^n}$ or all $x \in \mathbb{R}$ and $\sum\limits_{n=0}^{\infty} \frac{1}{2^n}$ is convergent

(geometric series with $r = \frac{1}{2}$), then by Weierstrass's criterion the function series $\sum\limits_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $\mathbb{R}$.

b) For all $n \in \mathbb{N}$ the function f_n is differentiable, there exists $x_0 \in \mathbb{R}$ where the numerical series $\sum\limits_{n=1}^{\infty} f_n(x_0)$ converges and the function series $\sum\limits_{n=1}^{\infty} f_n'(x) = \sum\limits_{n=1}^{\infty} \frac{n}{2^n} \cdot \frac{1}{1+(n\,x)^2}$ is also uniformly convergent on $\mathbb{R}$ by the Weierstrass criterion, since

$$|\, f_n'(x)\, | = \left| \frac{n}{2^n} \cdot \frac{1}{1+(n\,x)^2} \right| < \frac{n}{2^n} \text{ and } \sum_{n=1}^{\infty} \frac{n}{2^n} \text{ is convergent.}$$

The convergence of $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$ follows from the root test, since $\sqrt[n]{\frac{n}{2^n}} = \frac{\sqrt[n]{n}}{2} \longrightarrow \frac{1}{2} < 1$.

Therefore, $S'(x) = \frac{d}{dx}\left(\sum\limits_{n=1}^{\infty} \frac{\arctan(n\,x)}{2^n}\right) = \sum\limits_{n=1}^{\infty} \frac{d}{dx} \frac{\arctan(n\,x)}{2^n} = \sum\limits_{n=1}^{\infty} \frac{n}{2^n} \cdot \frac{1}{1+(n\,x)^2}$ for all $x \in \mathbb{R}$.

c) Using that $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$, if $|\, x\, | < 1$, differentiating term by term it follows that

$\left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=0}^{\infty} (x^n)' = \sum_{n=1}^{\infty} n\, x^{n-1} = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$, if $|x| < 1$. Multiplying both sides by x, we get

$\sum_{n=1}^{\infty} n\, x^n = \frac{x}{(1-x)^2}$, if $|x| < 1$. Substituting $x = \frac{1}{2}$ we obtain that $S'(0) = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2.$

4. (6+6 points) Find the Taylor series of the following functions at $x_0 = 3$ and determine the radius of convergence.

a) $f(x) = \frac{1}{x+7}$ **b)** $g(x) = e^{2x+1}$

Solution.

a) $f(x) = \frac{1}{(x-3)+10} = \frac{1}{10} \cdot \frac{1}{1-\left(-\frac{x-3}{10}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{n+1}} (x-3)^n$, where

$|r| = \left|-\frac{x-3}{10}\right| < 1 \Longrightarrow |x-3| < 10 \Longrightarrow R = 10.$

b) $g(x) = e^{4x+2} = e^{4(x-3)+14} = e^{14} \sum_{k=0}^{\infty} \frac{(4(x-3))^n}{n!} = \sum_{k=0}^{\infty} e^{14} \cdot \frac{4^n}{n!} (x-3)^n$ for all $x \in \mathbb{R} \Longrightarrow R = \infty$.

5. (6+6+6 points) Let $f(x) = \frac{1}{\sqrt{16+x^4}}$.

a) Find the Taylor series of f at $x_0 = 0$ and determine the radius of convergence.

b) Calculate $f^{(15)}(0)$ and $f^{(16)}(0)$.

c) Using part a), determine the approximate value of the integral $f(x) = \int_0^1 f(x)\, dx$ such that f is approximated by its Taylor polynomial of order 4. Give an estimation for the error.

Solution.

a) Using that $(1+u)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} u^k$, where $|u| < 1 = R$, the Taylor series of f is

$$f(x) = \frac{1}{\sqrt{16+x^4}} = \frac{1}{4} \cdot \frac{1}{\left(1+\frac{x^4}{16}\right)^{1/2}} = \frac{1}{4}\left(1+\frac{x^4}{16}\right)^{-\frac{1}{2}} = \frac{1}{4} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(\frac{x^4}{16}\right)^k = \sum_{k=0}^{\infty} \frac{1}{4} \binom{-\frac{1}{2}}{k} \frac{1}{16^k} x^{4k}$$

The radius of convergence: $|u| = \left|\frac{x^4}{16}\right| < 1 \Longrightarrow |x| < 2 \Longrightarrow R = 2$

b) For the values of the derivatives we use that

$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \Longrightarrow f^{(n)}(0) = n! \cdot a_n$, where a_n is the coefficient of x^n.

- To find the coefficient of x^{15} we have to solve $6k = 15$, where $k \in \mathbb{N}$.
  This equation doesn't have an integer solution, so $a_{15} = 0$
  (the term x^{15} is not included in the series)
  $\Longrightarrow f^{(15)}(0) = 15! \cdot a_{15} = 0$.

- The coefficient of x^{16}: $4k = 16$, where $k \in \mathbb{N} \Longrightarrow k = 4$

$$\Longrightarrow f^{(16)}(0) = 16!\cdot a_{16} = 16!\cdot\frac{1}{4}\binom{-\frac{1}{2}}{4}\frac{1}{16^4} = 16!\cdot\frac{1}{4}\cdot\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{16^4}.$$

c) Using part a), $f(x) = \sum_{k=0}^{\infty}\frac{1}{4}\binom{-\frac{1}{2}}{k}\frac{1}{16^k}x^{4k} = \frac{1}{4}\left(1+\binom{-\frac{1}{2}}{1}\frac{1}{16}x^4+\binom{-\frac{1}{2}}{2}\frac{1}{16^2}x^8+\ldots\right) =$

$= \frac{1}{4}\left(1-\frac{1}{2}\cdot\frac{1}{16}x^4+\frac{3}{8}\cdot\frac{1}{16^2}x^8+\ldots\right)$

$\Longrightarrow$ The Taylor polynomial of order 4 is $T_4(x) = \frac{1}{4}\left(1-\frac{1}{2}\cdot\frac{1}{16}x^4\right)$.

Since $[0, 1]\subset(-2, 2)$ then we can integrate term by term:

$\int_0^1 f(x)\,dx = \int_0^1\frac{1}{4}\left(1-\frac{1}{2}\cdot\frac{1}{16}x^4+\frac{3}{8}\cdot\frac{1}{16^2}x^8+\ldots\right)dx =$

$= \frac{1}{4}\left[x-\frac{1}{2}\cdot\frac{1}{16}\cdot\frac{x^5}{5}+\frac{3}{8}\cdot\frac{1}{16^2}\cdot\frac{x^9}{9}+\ldots\right]_0^1 = \frac{1}{4}\left(1-\frac{1}{2}\cdot\frac{1}{16}\cdot\frac{1}{5}+\frac{3}{8}\cdot\frac{1}{16^2}\cdot\frac{1}{9}+\ldots\right)\approx$

$\approx\frac{1}{4}\left(1-\frac{1}{2}\cdot\frac{1}{16}\cdot\frac{1}{5}\right)$. The error is $|E| < \frac{1}{4}\cdot\frac{3}{8}\cdot\frac{1}{16^2}\cdot\frac{1}{9}$ (Leibniz-series).

Remark: $\int_0^1\frac{1}{\sqrt{16+x^4}}\,dx\approx 0.248477$, $\int_0^1\left(\frac{1}{4}-\frac{x^4}{128}\right)dx\approx 0.248438$ and the error is at most 0.0000406901.

6. (4+6+5 points) Let $A=\{(x, y)\in\mathbb{R}^2 : 0\le x<1,\ |y| < x^2\}$.

a) Sketch the set A.

b) Find the set of interior points and the set of boundary points of A.

c) Find the set of limit points and the closure of A.

Solution.

b) The set of interior points of A: $\operatorname{int} A = A = \{(x, y)\in\mathbb{R}^2 : 0<x<1,\ |y| < x^2\}$

The set of boundary points of A: $\partial A = \{(x, y)\in\mathbb{R}^2 : 0\le x\le 1,\ |y| = x^2\}\cup\{(x, y)\in\mathbb{R}^2 : x=1,\ -1\le y\le 1\}$

c) The set of limit points of A: $A' = \{(x, y)\in\mathbb{R}^2 : 0\le x\le 1,\ |y| \le x^2\}$

The closure of A: $\operatorname{cl} A = A'$.

7. (8+7 points) Calculate the following limits if they exist.

a) $\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2+y^2}$ **b)** $\lim_{(x,y)\to(0,0)}\frac{\sin(xy)}{x^2+y^2}$

Solution.

a) Using polar coordinates, let $x = r\cos\varphi$, $y = r\sin\varphi$, where $r\longrightarrow 0$ and φ is arbitrary.

Then the limit at the origin is

$$\lim_{r\to 0}\frac{r^2\cos^2\varphi\, r^2\sin^2\varphi}{r^2\cos^2\varphi+r^2\sin^2\varphi} = \lim_{r\to 0}\frac{r^4\cos^2\varphi\sin^2\varphi}{r^2} = \lim_{r\to 0}r^2\cos^2\varphi\sin^2\varphi = 0,$$

since $r^2\longrightarrow 0$ and $\cos^2\varphi\sin^2\varphi$ is bounded. So $\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2+y^2} = 0$.

b) Also with polar coordinates, the limit at the origin is

$$\lim_{r\to 0} \frac{\sin(r^2 \cos\varphi \sin\varphi)}{r^2} \lim_{r\to 0} \frac{\sin(r^2 \cos\varphi \sin\varphi)}{r^2 \cos\varphi \sin\varphi} \cdot \cos\varphi \sin\varphi = 1 \cdot \cos\varphi \sin\varphi.$$

Since the above limit depends on φ, then the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x\,y)}{x^2+y^2}$ doesn't exist.

8.* (10 points - BONUS)

Prove that $\sum\limits_{n=1}^{\infty}\left(1-\cos\dfrac{x}{n}\right)$ converges uniformly and is differentiable on any bounded interval $(a, b) \subset \mathbb{R}$. (Help: use Lagrange's mean value theorem to get a bound for $\left|1-\cos\dfrac{x}{n}\right|$.)

Solution. We apply Lagrange's mean value theorem for the function $f(x) = \cos\dfrac{x}{n}$ on the interval $[0, x]$, where $x \in (a, b)$. For all $x \in \mathbb{R}$ there exists c between 0 and x such that $f'(c) = \dfrac{f(0)-f(x)}{0-x} \Longrightarrow$

$$\left|1-\cos\frac{x}{n}\right| = \left|\cos\frac{0}{n} - \cos\frac{x}{n}\right| = \left|\frac{1}{n}\sin\frac{c}{n}\right| \cdot |x|.$$

Using that $|\sin\alpha| \leq |\alpha|$ we obtain $\left|1-\cos\dfrac{x}{n}\right| \leq \dfrac{|c\,x|}{n^2} \leq \dfrac{M^2}{n^2}$, where $M = \max\{|a|, |b|\}$.

Since $\sum\limits_{n=1}^{\infty}\dfrac{M^2}{n^2}$ converges, then the series $\sum\limits_{n=1}^{\infty}\left(1-\cos\dfrac{x}{n}\right)$ is uniformly convergent on (a, b) by the Weierstrass criterion.

Differentiating term by term, we obtain the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{x}{n}$.

Since $\left|\dfrac{1}{n}\sin\dfrac{x}{n}\right| \leq \dfrac{|x|}{n^2} \leq \dfrac{M}{n^2}$ for $x \in (a, b)$ where M is defined above, then by the Weierstrass criterion the function series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{x}{n}$ also converges uniformly on (a, b).

From this it follows that the original series is differentiable on (a, b).