Calculus 2, Sample Test 2

1. Let $f(x, y) = \begin{cases} \dfrac{x^3 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

a) Where is f continuous?

b) Calculate the partial derivatives of f where they exist.

c) Where is f differentiable?

2. Let $f(x, y) = \sqrt{2x^2 + y^2}$ and

a) Where does the gradient of f exist?

b) Find the equation f the tangent line of f at the point $P(2, 1)$.

b) Find the directional derivative of f at $P(2, 1)$ in the direction $\boldsymbol{v} = (3, 4)$.

c) Find the directional derivative of f at $P(2, 1)$ with the maximal value.

3. Determine the Taylor polynomial of order 2 of the function $f(x, y) = (1 + e^y) \cos x - y e^x$ at the point $P(0, 1)$.

4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = \left(2x^3 y - x y + x y^2, x \ln(3x - y) - 3y^2\right)$. Plug in $(x, y) = (1, 2)$ to obtain $f(1, 2) = (6, -12)$. Show that in a small neighbourhood of $(6, -12)$ the inverse function f^{-1} exists and determine the derivative $\left(f^{-1}\right)'(6, -12)$.

5. Determine the local minima and maxima of the function $f(x, y) = x + \dfrac{y^2}{4x} + \dfrac{1}{y}$

6. Determine the maximum and minimum of the function $f(x, y) = x + y$ under the constraint $x^4 + y^4 = 2$.

Additional exercises

7. Determine the maximum and minimum of the function $f(x, y, z) = x^3 + y^2 + z$ under the constraint $x^2 + y^2 + z^2 = 1$

8. Let $\alpha > 1$ and consider the function $f(x, y) = \begin{cases} \left(x^2 + y^2\right)^{\frac{\alpha}{2}} \cdot \sin \dfrac{1}{\left(x^2 + y^2\right)^{\frac{\beta}{2}}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

a) Calculate the first-order partial derivatives of f where they exist.

b) Prove that f is continuously differentiable on the set $\mathbb{R}^2 \setminus \{(0, 0)\}$.

c) Are the partial derivatives continuous at the origin?

d) Using the definition of differentiability, decide whether f is differentiable at the origin.

Solutions

1. Let $f(x, y) = \begin{cases} \dfrac{x^3 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

a) Where is f continuous?

b) Calculate the partial derivatives of f where they exist. (At the origin use the definition.)

c) Where is f differentiable?

Solution.

a) Outside of the origin f is continuous, since it is the ratio of two continuous functions, and the denominator is not zero.

b) At the origin we calculate the limit on the straight lines $y = m\,x$:

$\lim\limits_{x \to 0} f(x, m\,x) = \lim\limits_{x \to 0} \dfrac{x^3 - m^2 x^2}{x^2 + m^2 x^2} = -\dfrac{m^2}{1 + m^2}$. It depends on m

$\Longrightarrow \lim\limits_{(x,y) \to (0,0)} f(x, y)$ doesn't exist $\Longrightarrow$ f is not continuous at the origin.

c) If $(x, y) \neq (0, 0)$ then

$$f'_x(x, y) = \frac{3\,x^2 (x^2 + y^2) - (x^3 - y^2) \cdot 2\,x}{(x^2 + y^2)^2}, \quad f'_y(x, y) = \frac{-2\,y (x^2 + y^2) - (x^3 - y^2) \cdot 2\,y}{(x^2 + y^2)^2}$$

If $(x, y) = (0, 0)$ then

$$f'_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2}}{h} = 1,$$

$$f'_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{-\frac{h^2}{h^2}}{h} = \lim_{h \to 0} \frac{-1}{h}, \text{ which doesn't exist.}$$

d) Outside of the origin the partial derivatives are continuous, so f is differentiable on this open set.
At the origin f is not differentiable, since it is not continuous.
Or: f is not differentiable at the origin, since $f'_y(0, 0)$ doesn't exist.

2. Let $f(x, y) = \sqrt{2\,x^2 + y^2}$ and

a) Where does the gradient of f exist?

b) Find the equation f the tangent line of f at the point $P(2, 1)$.

b) Find the directional derivative of f at $P(2, 1)$ in the direction $\boldsymbol{v} = (3, 4)$.

c) Find the directional derivative of f at $P(2, 1)$ with the maximal value.

Solution.

a) The partial derivatives are: $f'_x(x, y) = \dfrac{2\,x}{\sqrt{2\,x^2 + y^2}}$, $f'_y(x, y) = \dfrac{y}{\sqrt{2\,x^2 + y^2}}$

The gradient of f exist outside of the origin, since then the partial derivatives exist and are continuous. If $(x, y) \neq (0, 0)$ then $\operatorname{grad} f(x, y) = \left(f'_x(x, y), f'_y(x, y)\right)$.

b) The equation of the tangent plane is $z = f(a, b) + f'_x(a, b)(x - a) + f'_y(a, b)(y - b)$.

Here $(a, b) = (2, 1) \Longrightarrow f(2, 1) = 3,\ f'_x(2, 1) = \frac{4}{3},\ f'_y(2, 1) = \frac{1}{3}$

$\Longrightarrow$ the equation of the tangent plane is $z = 3 + \frac{4}{3}(x - 2) + \frac{1}{3}(y - 1)$.

c) Since $\|\boldsymbol{v}\| = \sqrt{3^2 + 4^2} = 5$ then the unit vector parallel to $\boldsymbol{v}$ is $\boldsymbol{e} = \frac{1}{\|\boldsymbol{v}\|} \cdot \boldsymbol{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$.

The gradient vector of f at $P(2, 1)$ is $\operatorname{grad} f(2, 1) = \left(\frac{4}{3}, \frac{1}{3}\right)$.

The directional derivative of f at $P(2, 1)$ in the direction $\boldsymbol{v} = (3, 4)$ is

$\boldsymbol{e} \cdot \operatorname{grad} f(2, 1) = < \left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{4}{3}, \frac{1}{3}\right) > = \frac{3}{5} \cdot \frac{4}{3} + \frac{4}{5} \cdot \frac{1}{3} = \frac{16}{15}$

d) The directional derivative of f is maximal in the direction $\frac{\operatorname{grad} f(2, 1)}{\|\operatorname{grad} f(2, 1)\|}$ and the value of the maximum is $\|\operatorname{grad} f(2, 1)\| = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{17}}{3}$.

3. Determine the Taylor polynomial of order 2 of the function $f(x, y) = (1 + e^y)\cos x - y\, e^x$ at the point $P(0, 1)$.

Solution. Calculate the first-order and second order partial derivatives of f and evaluate them at the given point:

$f(x, y) = (1 + e^y)\cos x - y\, e^x \quad \Longrightarrow f(0, 1) = (1 + e) \cdot 1 - 1 \cdot 1 = e$

$f'_x(x, y) = -(1 + e^y)\sin x - y\, e^x \quad \Longrightarrow f'_x(0, 1) = -(1 + e) \cdot 0 - 1 \cdot 1 = -1$

$f'_y(x, y) = e^y \cos x - e^x \quad \Longrightarrow f'_y(0, 1) = e \cdot 1 - 1 = e - 1$

$f''_{xx}(x, y) = -(1 + e^y)\cos x - y\, e^x \quad \Longrightarrow f''_{xx}(0, 1) = -(1 + e) \cdot 1 - 1 \cdot 1 = -e - 2$

$f''_{xy}(x, y) = -e^y \sin x - e^x \quad \Longrightarrow f''_{xy}(0, 1) = -e \cdot 0 - 1 = -1$

$f''_{yx}(x, y) = -e^y \sin x - e^x \quad \Longrightarrow f''_{yx}(x, y) = -1$

$f''_{yy}(x, y) = e^y \cos x \quad \Longrightarrow f''_{yy}(0, 1) = e \cdot 1 = e$

The Taylor polynomial of order 2 at a point (a, b) is

$T_2(x, y) = f(a, b) + f'_x(a, b)(x - a) + f'_y(a, b)(y - b) +$

$$+ \frac{1}{2}\left(f''_{xx}(a, b)(x - a)^2 + 2 f''_{xy}(a, b)(x - a)(x - b) + f''_{yy}(a, b)(y - b)^2\right)$$

Substituting $(a, b) = (0, 1)$:

$f(x, y) \approx T_2(x, y) = e + (-1)(x - 0) + (e - 1)(y - 1) + \frac{1}{2}\left((-e - 2)(x - 0)^2 + 2(-1)(x - 0)(y - 1) + e(y - 1)^2\right)$

4. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = \left(2x^3 y - x y + x y^2,\ x \ln(3x - y) - 3y^2\right)$. Plug in $(x, y) = (1, 2)$ to obtain $f(1, 2) = (6, -12)$. Show that in a small neighbourhood of $(6, -12)$ the inverse function f^{-1} exists and determine the derivative $\left(f^{-1}\right)'(6, -12)$.

Solution. $f(1, 2) = (2\cdot 1\cdot 2 - 1\cdot 2 + 1\cdot 4,\ 1\cdot\ln(1) - 3\cdot 4) = (6, -12)$.
Calculating the Jacobian matrix of f :

$$f'(x, y) = \begin{pmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 6x^2 y - y + y^2 & 2x^3 - x + 2xy \\ \ln(3x - y) + \frac{3x}{3x - y} & \frac{-x}{3x - y} - 6y \end{pmatrix}$$

Substituting $(x, y) = (1, 2)$ into the Jacobian:

$$f'(1, 2) = \begin{pmatrix} 14 & 5 \\ 3 & -13 \end{pmatrix}$$

Calculating the determinant of the Jacobian:
$$\det f'(1, 2) = \det\begin{pmatrix} 14 & 5 \\ 3 & -13 \end{pmatrix} = -14\cdot 13 - 15 \neq 0$$

Since the above determinant is not zero then by the inverse function theorem,
the inverse function f^{-1} exists in a small neighbourhood if the point $f(1, 2) = (6, -12)$.

The derivative of f^{-1} at $(6, -12)$ is the inverse of $f'(1, 2)$. Using that
the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, then
$$\left(f^{-1}\right)'(6, -12) = \left(f'(1, 2)\right)^{-1} = \frac{1}{-14\cdot 13 - 15}\begin{pmatrix} -13 & -5 \\ -3 & 14 \end{pmatrix} = \frac{1}{197}\begin{pmatrix} 13 & 5 \\ 3 & -14 \end{pmatrix}.$$

5. Determine the local minima and maxima of the function $f(x, y) = x + \frac{y^2}{4x} + \frac{1}{y}$

Solution. Here $x \neq 0$, $y \neq 0$. The first-order partial derivatives of f are:

(1) $f'_x(x, y) = 1 - \frac{y^2}{4x^2} = 0 \quad\Longrightarrow\quad 4x^2 = y^2 \quad\Longrightarrow\quad 2x = \pm y$

(2) $f'_y(x, y) = \frac{y}{2x} - \frac{1}{y^2} = 0 \quad\Longrightarrow\quad y^3 = 2x$

Case 1. If $2x = y$ then $y^3 = y \quad\Longrightarrow\quad y^3 - y = y(y - 1)(y + 1) = 0$

Since $y \neq 0$ then $y_1 = 1,\ y_2 = -1 \quad\Longrightarrow\quad x_1 = \frac{1}{2},\ x_2 = -\frac{1}{2}$

Case 2. If $2x = -y$ then $y^3 = -y \Longrightarrow y\left(y^2 + 1\right) = 0$.

This cannot be the case, since $y \neq 0$ and $y^2 + 1 > 0$.

The stationary points are: $P_1\left(\frac{1}{2}, 1\right)$ and $P_1\left(-\frac{1}{2}, -1\right)$.

The Hesse-matrix of f is $H(x, y) = \begin{pmatrix} f''_{xx}(x, y) & f''_{xy}(x, y) \\ f''_{yx}(x, y) & f''_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} \frac{y^2}{2x^3} & -\frac{y}{2x^2} \\ -\frac{y}{2x^2} & \frac{1}{2x} + \frac{2}{y^3} \end{pmatrix}$

Evaluating the Hesse-matrix at the given points:

At $P_1\left(\frac{1}{2}, 1\right)$: $H(P_1) = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$. Since $\det H(P_1) = 12 - 4 = 8 > 0$ and $f''_{xx}(P_1) = 4 > 0$ then f has a local minimum at $P_1\left(\frac{1}{2}, 1\right)$ with $f\left(\frac{1}{2}, 1\right) = 2$.

At $P_2\left(-\frac{1}{2}, -1\right)$: $H(P_2) = \begin{pmatrix} -4 & 2 \\ 2 & -3 \end{pmatrix}$. Since $\det H(P_2) = 12 - 4 = 8 > 0$ and $f''_{xx}(P_2) = -4 < 0$ then f has a local maximum at $P_2\left(-\frac{1}{2}, -1\right)$ with $f\left(-\frac{1}{2}, -1\right) = -2$.

Remark. We can avoid calculating the Hesse matrix with the following observation:
a) $f(-x, -y) = -f(x, y)$, therefore, if f has a local minimum at (x_0, y_0) then f has a local maximum at $(-x_0, -y_0)$.
b) If $x > 0$, $y > 0$, then $f(x, y) > 0$, and thus f is bounded below.
c) The limit of $f(x, y)$ is infinity if

- $x \longrightarrow \infty$, y is fixed, or
- $y \longrightarrow \infty$, x is fixed, or
- $x \longrightarrow 0$, y is fixed, or
- $y \longrightarrow 0$, x is fixed.

Therefore, f can only have a local minimum on the region $x > 0$, $y > 0$.
So f has a local minimum at P_1 and a local maximum at P_2.

6. Determine the maximum and minimum of the function $f(x, y) = x + y$ under the constraint $x^4 + y^4 = 2$.

Solution. The constraint is given by the function $g(x, y) = x^4 + y^4 - 2$. Applying the Lagrange multiplier method,
consider the function $L(x, y) = f(x, y) + \lambda\, g(x, y) = x + y + \lambda\left(x^4 + y^4 - 2\right)$
The first-order partial derivatives of L are:

$$L'_x = 1 + 4x^3 \cdot \lambda = 0 \quad \Longrightarrow \quad x^3 = -\frac{1}{4\lambda}$$

$$L'_y = 1 + 4y^3 \cdot \lambda = 0 \quad \Longrightarrow \quad y^3 = -\frac{1}{4\lambda}$$

$$g(x, y) = x^4 + y^4 - 2 = 0 \quad \Longrightarrow \quad \left(\sqrt[3]{-\frac{1}{4\lambda}}\right)^4 + \left(\sqrt[3]{-\frac{1}{4\lambda}}\right)^4 = 2\left(\sqrt[3]{-\frac{1}{4\lambda}}\right)^4 = 2$$

$$\Longrightarrow \quad \sqrt[3]{-\frac{1}{4\lambda}} = \pm 1 \quad \Longrightarrow \quad -\frac{1}{4\lambda} = \pm 1 \quad \Longrightarrow \quad \lambda = \pm\frac{1}{4}$$

If $\lambda = \frac{1}{4}$ then $x = y = -1 \Longrightarrow P_1(-1, -1)$

and if $\lambda = -\frac{1}{4}$ then $x = y = 1 \Longrightarrow P_2(1, 1)$

The second order partial derivatives of L are:
$L''_{xx} = 12x^2 \cdot \lambda$, $L''_{xy} = 0$, $L''_{yy} = 12y^2 \cdot \lambda$

The Hesse-matrix of L is $H(x, y) = \begin{pmatrix} L''_{xx}(x, y) & L''_{xy}(x, y) \\ L''_{yx}(x, y) & L''_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} 12x^2 \cdot \lambda & 0 \\ 0 & 12y^2 \cdot \lambda \end{pmatrix}$

• If $\lambda = \frac{1}{4}$ then $H(-1, -1) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

Since $\det H(-1, -1) = 9 > 0$ and $f''_{xx}(-1, -1) = 3 > 0$ then f has a local minimum at $P_1(-1, -1)$.

• If $\lambda = -\frac{1}{4}$ then $H(1, 1) = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$.

Since $\det H(1, 1) = 9 > 0$ and $f''_{xx}(1, 1) = -3 < 0$ then f has a local maximum at $P_1(1, 1)$.

Remark. Since the constraint defines the surface of a sphere then by Weierstrass's min-max theorem f has a minimum and a maximum with this constraint. Substituting the coordinates of the stationary points, we obtain that $f(1, 1) = 2$ is the maximum and $f(-1, -1) = -2$ is the minimum.

7. Determine the maximum and minimum of the function $f(x, y, z) = x^3 + y^2 + z$ under the constraint $x^2 + y^2 + z^2 = 1$

Solution. Substituting $y^2 = 1 - x^2 - z^2$ into $f(x, y, z)$, we obtain the function
$g(x, z) = x^3 - x^2 - z^2 + z + 1$

The first-order partial derivatives of g are:

(1) $g'_x(x, z) = 3x^2 - 2x = x(3x - 2) = 0 \quad \Longrightarrow \quad x_1 = 0,\ x_2 = \frac{2}{3}$

(2) $g'_z(x, z) = -2z + 1 = 0 \quad \Longrightarrow \quad z = \frac{1}{2}$

The stationary points are: $P_1\left(0, \frac{1}{2}\right)$ and $P_1\left(\frac{2}{3}, \frac{1}{2}\right)$.

The Hesse-matrix of g is $H(x, y) = \begin{pmatrix} g''_{xx}(x, y) & g''_{xy}(x, y) \\ g''_{yx}(x, y) & g''_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} 6x - 2 & 0 \\ 0 & -2 \end{pmatrix}$

Evaluating the Hesse-matrix at the given points:

At $P_1\left(0, \frac{1}{2}\right)$: $H(P_1) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$. Since $\det H(P_1) = 4 > 0$ and $f''_{xx}(P_1) = -2 < 0$ then f has a local maximum at $P_1\left(0, \frac{1}{2}\right)$.

At $P_2\left(\frac{2}{3}, \frac{1}{2}\right)$: $H(P_2) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$. Since $\det H(P_2) = -4 < 0$ then P_2 is a saddle point.

At $(x, z) = \left(0, \frac{1}{2}\right)$, from the condition $x^2 + y^2 + z^2 = 1$ we obtain $y = \pm\frac{\sqrt{3}}{2}$.

Therefore, f has a local maximum at the points $\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(0, -\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and the value of the maximum is $\frac{5}{4}$.

8. Let $\alpha > 1$ and consider the function $f(x, y) = \begin{cases} (x^2 + y^2)^{\frac{\alpha}{2}} \cdot \sin \dfrac{1}{(x^2 + y^2)^{\frac{\beta}{2}}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

a) Calculate the first-order partial derivatives of f where they exist.
b) Prove that f is continuously differentiable on the set $\mathbb{R}^2 \backslash \{(0, 0)\}$.
c) Are the partial derivatives continuous at the origin?
d) Using the definition of differentiability, decide whether f is differentiable at the origin.

Solution. If $(x, y) \neq (0, 0)$ then

$$f'_x(x, y) = \alpha x (x^2 + y^2)^{\frac{\alpha}{2}-1} \cdot \sin \frac{1}{(x^2 + y^2)^{\frac{\beta}{2}}} - \beta x (x^2 + y^2)^{\frac{\alpha}{2}-\frac{\beta}{2}-1} \cdot \cos \frac{1}{(x^2 + y^2)^{\frac{\beta}{2}}}.$$

At the origin using the definition:

$$f'_x(0, 0) = \lim_{h \to 0} \frac{(|h|)^{\alpha} \cdot \sin \frac{1}{|h|^{\beta}}}{h} = 0.$$

$f'_y(x, y)$ and $f'_y(x, y)$ can be obtained from these by the changing the variables.

b) The partial derivatives are continuous on $\mathbb{R}^2 \backslash \{(0, 0)\}$, since they are compositions of continuous functions, therefore f is continuously differentiable on this open set.
c) Since f is symmetric in the variables then it is enough to investigate the partial derivative f'_x.
It is continuous in the origin if and only if $\lim\limits_{(x,y)\to(0,0)} f'_x(x, y) = f'_x(0, 0)$.

We calculate the limit using polar coordinates: $x = r \cos \varphi$, $y = r \sin \varphi$:

$$\lim_{(x,y)\to(0,0)} f'_x(x, y) = \lim_{r\to 0+} \left(\alpha r \cos \varphi (r^2)^{\frac{\alpha}{2}-1} \cdot \sin \frac{1}{r} - \beta r \cos \varphi (r^2)^{\frac{\alpha}{2}-\frac{\beta}{2}-1} \cdot \cos \frac{1}{r^{\beta}} \right) =$$

$$= \lim_{r\to 0+} \left(\alpha \cos \varphi \cdot r^{\alpha-1} \cdot \sin \frac{1}{r} - \beta \cos \varphi \cdot r^{\alpha-\beta-1} \cdot \cos \frac{1}{r^{\beta}} \right)$$

$$= 0 - \beta \cos \varphi \cdot \lim_{r\to 0+} r^{\alpha-\beta-1} \cdot \cos \frac{1}{r^{\beta}} = \begin{cases} 0 & \text{if } \alpha > \beta + 1 \\ \text{doesn't exist} & \text{if } \alpha \leq \beta + 1 \end{cases}$$

Therefore, the partial derivatives are continuous if and only if $\alpha > \beta + 1$.
d) If f is differentiable at the origin then its derivative can only be the mapping $A = (0 \quad 0)$.

Since $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x, y) - f(0, 0) - A(x, y)}{\|(x, y)\|} = \lim\limits_{r\to 0+} \dfrac{r^{\alpha} \sin \frac{1}{r^{\beta}}}{r} = 0$,
then f is differentiable at the origin.