Calculus 2, Sample Test 1

1. a) Calculate the value of the following integral: $\displaystyle\int_0^{\infty} x\, e^{-x}\, dx$

b) Decide whether the following integral converges or diverges: $\displaystyle\int_0^1 \frac{\sin(x^2+1)}{(1+x)\sqrt{x}}\, dx$

2. Let $f_n(x) = \dfrac{e^{nx}-1}{e^{2nx}+1}$. Determine the pointwise limit of $f_n(x)$ on the real line $\mathbb{R}$ and decide whether the convergence is uniform or not.

3. a) Show that the series $S(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\arctan(\frac{x}{n})}{n^2}$ is convergent for all $x \in \mathbb{R}$.

b) Show that $S'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + n x^2}$ for all $x \in \mathbb{R}$.

4. Estimate the value of $\sqrt[3]{1.2}$ by the Taylor polynomial of order 2 of $f(x) = \sqrt[3]{1+x}$ at center 0. Give an upper bound for the error of the approximation.

5. Find the Taylor series of the following functions at $x_0 = 1$ and determine the radius of convergence.

a) $f(x) = \dfrac{1}{x+3}$ **b)** $g(x) = \dfrac{1}{(x+3)^2}$

6. Find the Taylor series of the function $f(x) = \dfrac{2x^3}{\sqrt[5]{32-2x^2}}$ at $x_0 = 0$ and determine the radius of convergence. Calculate $f^{(102)}(0)$ and $f^{(103)}(0)$.

7. Let $A = \{(x, y) \in \mathbb{R}^2 : x \le 0,\ x^2 + y^2 < 1\} \cup \{(x, 0) \in \mathbb{R}^2 : -1 \le x \le 1\}$.
Find the set of interior points, the set of limit points and the closure of A.

8. Calculate the following limits if they exist.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\alpha x^2 y + \beta y^2}{x^4 + y^2}$ **b)** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^{2\beta} y + \sin(x^2+y^2)^{\alpha}}{(x^2+y^2)^{\beta}}$

Calculus 2, Sample Test 1, solutions

1. a) Calculate the value of the following integral: $\int_0^\infty x\, e^{-x}\, dx$

b) Decide whether the following integral converges or diverges: $\int_0^1 \frac{\sin(x^2+1)}{(1+x)\sqrt{x}}\, dx$

Solution. a) First we calculate the indefinite integral using the integration by parts method:

$\int x\, e^{-x}\, dx = -x\, e^{-x} - \int -e^{-x}\, dx = -x\, e^{-x} - e^{-x} + c.$

The improper integral: $\int_0^\infty x\, e^{-x}\, dx = \lim_{A\to\infty} \int_0^A x\, e^{-x}\, dx = \lim_{A\to\infty} [-x\, e^{-x} - e^{-x}]_0^A =$
$\lim_{A\to\infty} \left(\left(-A\, e^{-A} - e^{-A}\right) - \left(0 - e^0\right)\right) = (0-0) - \left(0 - e^0\right) = 1$

Here we use that $\lim_{A\to\infty} e^{-A} = 0$ and $\lim_{A\to\infty} \left(A\, e^{-A}\right) = \lim_{A\to\infty} \frac{A}{e^A} \overset{\frac{\infty}{\infty},\, L'H}{=} \lim_{A\to\infty} \frac{1}{e^A} = \frac{1}{\infty} = 0.$

b) If $x \in [0, 1]$ then $0 \le f(x) = \frac{\sin(x^2+1)}{(1+x)\sqrt{x}} \le \frac{1}{(1+0)\sqrt{x}} = \frac{1}{\sqrt{x}} = g(x)$

and the improper integral of g is convergent on $[0, 1]$,

since $\int_0^1 g(x)\, dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 x^{-\frac{1}{2}}\, dx = \lim_{\varepsilon\to 0} \left[2\sqrt{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0} \left(2 - 2\sqrt{\varepsilon}\right) = 2,$

so the improper integral $\int_0^1 \frac{\sin(x^2+1)}{(1+x)\sqrt{x}}\, dx$ converges.

2. Let $f_n(x) = \frac{e^{nx} - 1}{e^{2nx} + 1}$. Determine the pointwise limit of $f_n(x)$ on the real line $\mathbb{R}$ and decide whether the convergence is uniform or not.

Solution. If $x > 0$, then $\lim_{n\to\infty} e^{nx} = \infty$, so $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1 - \frac{1}{e^{nx}}}{e^{nx} + \frac{1}{e^{nx}}} = \lim_{n\to\infty} \frac{1-0}{\infty + 0} = 0.$

If $x = 0$, then $e^{nx} = 1$, so $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1-1}{1+1} = 0.$

If $x < 0$, then $\lim_{n\to\infty} e^{nx} = 0$, so $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{0-1}{0+1} = -1.$

Therefore, the limit function is $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } x \ge 0 \\ -1, & \text{if } x < 0 \end{cases}.$

The convergence of (f_n) to f on $\mathbb{R}$ is not uniform, since f has a discontinuity at $x = 0$, but f_n is continuous on $\mathbb{R}$ for all $n \in \mathbb{N}$.

Or: We can use that if there exists a sequence $(x_n) \subset H$ such that
$\lim_{n\to\infty} |r_n(x_n)| = \lim_{n\to\infty} |f_n(x_n) - f(x_n)| \ne 0$, then (f_n) does not converge uniformly to f on H.

Here let $x_n = \frac{1}{n}$, then $\lim\limits_{n\to\infty} \mid r_n(x_n) \mid = \lim\limits_{n\to\infty} \mid f_n(x_n) - f(x_n) \mid = \lim\limits_{n\to\infty} \left| \frac{e^{n\cdot\frac{1}{n}} - 1}{e^{2n\cdot\frac{1}{n}} + 1} - 0 \right| = \frac{e-1}{e^2+1} \neq 0$

$\Longrightarrow$ (f_n) does not converge uniformly to f on $\mathbb{R}$.

3. a) Show that the series $S(x) = \sum\limits_{n=1}^{\infty} \frac{\arctan\left(\frac{x}{n}\right)}{n^2}$ is convergent for all $x \in \mathbb{R}$.

b) Show that $S'(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n^3 + n x^2}$ for all $x \in \mathbb{R}$.

Solution.

a) Since $\mid f_n(x) \mid = \left| \frac{\arctan\left(\frac{x}{n}\right)}{n^2} \right| < \frac{\pi}{2} \cdot \frac{1}{n^2}$ or all $x \in \mathbb{R}$ and $\sum\limits_{n=1}^{\infty} \frac{\pi}{2} \cdot \frac{1}{n^2}$ is convergent (p-series with $p = 2$),

then by Weierstrass's criterion the function series $\sum\limits_{n=1}^{\infty} \frac{\arctan\left(\frac{x}{n}\right)}{n^2}$ is uniformly convergent on $\mathbb{R}$.

b) The function series $\sum\limits_{n=1}^{\infty} f_n'(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} \cdot \frac{1}{1 + \left(\frac{x}{n}\right)^2} \cdot \frac{1}{n} = \sum\limits_{n=1}^{\infty} \frac{1}{n^3 + n x^2}$ is also uniformly convergent on $\mathbb{R}$

by the Weierstrass criterion, since $\mid f_n'(x) \mid = \left| \frac{1}{n^3 + n x^2} \right| < \frac{1}{n^3}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ is convergent,

so $S'(x) = \frac{d}{dx}\left(\sum\limits_{n=1}^{\infty} \frac{\arctan\left(\frac{x}{n}\right)}{n^2}\right) = \sum\limits_{n=1}^{\infty} \frac{d}{dx} \frac{\arctan\left(\frac{x}{n}\right)}{n^2} = \sum\limits_{n=1}^{\infty} \frac{1}{n^3 + n x^2}$ for all $x \in \mathbb{R}$.

4. Estimate the value of $\sqrt[3]{1.2}$ by the Taylor polynomial of order 2 of $f(x) = \sqrt[3]{1+x}$ at center 0. Give an upper bound for the error of the approximation.

Solution.

$f(x) = \sqrt[3]{1+x}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{3(1+x)^{2/3}}$ $\qquad f'(0) = \frac{1}{3}$

$f''(x) = -\frac{2}{9(1+x)^{5/3}}$ $\qquad f''(0) = -\frac{2}{9}$

$f'''(x) = \frac{10}{27(1+x)^{8/3}}$

The Taylor polynomial of order 2:

$$T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 + \frac{1}{3}x - \frac{2}{9\cdot 2!}x^2$$

If $x = 0.2$ then $\sqrt[3]{1.2} \approx T_2(0.2) = 1 + \frac{1}{2}\cdot 0.2 - \frac{2}{9\cdot 2!}0.2^2 \approx 1.06222$

Lagrange remainder term: $R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x - x_0)^3$, where $x_0 = 0$, $x = 0.2$, $0 < \xi < 0.2$

The error estimation:

$\mid E \mid = \mid R_2(x) \mid = \left| \frac{10}{27(1+\xi)^{8/3}} \cdot \frac{1}{3!}(0.2-0)^3 \right| =$

$$= \frac{10}{27\,(1+\xi)^{8/3}}\,\frac{1}{3!}\,0.2^3 < \frac{10}{27\,(1+0)^{8/3}}\,\frac{1}{3!}\,0.2^3 \approx 0.000493827$$

5. Find the Taylor series of the following functions at $x_0 = 1$ and determine the radius of convergence.

a) $f(x) = \frac{1}{x+3}$ **b)** $g(x) = \frac{1}{(x+3)^2}$

Solution.

a) $f(x) = \frac{1}{(x-1)+4} = \frac{1}{4}\cdot\frac{1}{1-\left(-\frac{x-1}{4}\right)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\,(x-1)^n$, where

$$|\,r\,| = \left|\,-\frac{x-1}{4}\,\right| < 1 \implies |\,x-1\,| < 4 \implies R_1 = 4.$$

b) $g(x) = -f\,'(x) = -\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\,(x-1)^n\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}\,\frac{d}{dx}\,(x-1)^n = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot n}{4^{n+1}}\,(x-1)^{n-1}$

because of uniform convergence if $|\,x-1\,| < 4$, so the radius of convergence of the series is $R_2 = 4$.

6. Find the Taylor series of the function $f(x) = \frac{2\,x^3}{\sqrt[5]{32-2\,x^2}}$ at $x_0 = 0$ and determine the radius of convergence. Calculate $f^{(102)}(0)$ and $f^{(103)}(0)$.

Solution. Let $g(x) = \frac{1}{\sqrt[5]{32-2\,x^2}}$. First we determine the Taylor series of g at $x_0 = 0$ using that

$(1+u)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}u^k$, where $|\,u\,| < 1 = R$.

$$\implies g(x) = \frac{1}{\sqrt[5]{32}}\cdot\frac{1}{\left(1+\left(-\frac{x^2}{16}\right)\right)^{1/5}} = \frac{1}{2}\left(1+\left(-\frac{x^2}{16}\right)\right)^{-\frac{1}{5}} = \frac{1}{2}\sum_{k=0}^{\infty}\binom{-\frac{1}{5}}{k}\left(-\frac{x^2}{16}\right)^k = \sum_{k=0}^{\infty}\frac{1}{2}\binom{-\frac{1}{5}}{k}\frac{(-1)^k}{16^k}\,x^{2k}$$

The radius of convergence: $|\,u\,| = \left|\,-\frac{x^2}{16}\,\right| < 1 \implies |\,x\,| < 4 \implies R = 4$

Using this, the Taylor series of f is

$f(x) = 2\,x^3\cdot g(x) = 2\,x^3\cdot\sum_{k=0}^{\infty}\frac{1}{2}\binom{-\frac{1}{5}}{k}\frac{(-1)^k}{16^k}\,x^{2k} = \sum_{k=0}^{\infty}\binom{-\frac{1}{5}}{k}\frac{(-1)^k}{16^k}\,x^{2k+3}$, where $R = 4$ (the same).

To find the values of the derivatives we use that

$g(x) = \sum_{n=0}^{\infty}a_n\,x^n = \sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}\,x^n \implies g^{(n)}(0) = n!\cdot a_n$, where a_n is the coefficient of x^n.

- To find the coefficient of x^{102} we have to solve $2\,k+3 = 102$, where $k \in \mathbb{N}$.
  This equation doesn't have an integer solution, so $a_{102} = 0$
  (the term x^{102} is not included in the series)
  $\implies g^{(102)}(0) = 102!\cdot a_{102} = 0.$

- The coefficient of x^{102}: $2k+3=103$, where $k\in\mathbb{N} \implies k=50$

  $\implies g^{(103)}(0) = 103!\cdot a_{103} = 103!\cdot\binom{-\frac{1}{5}}{50}\dfrac{(-1)^{50}}{16^{50}}$.

7. Let $A=\{(x,y)\in\mathbb{R}^2 : x\le 0,\ x^2+y^2<1\}\cup\{(x,0)\in\mathbb{R}^2 : -1\le x\le 1\}$.
Find the set of interior points, the set of limit points and the closure of A.

Solution.
The set of interior points of A: $\operatorname{int} A = A = \{(x,y)\in\mathbb{R}^2 : x<0,\ x^2+y^2<1\}$
The set of limit points of A: $A' = \{(x,y)\in\mathbb{R}^2 : x\le 0,\ x^2+y^2\le 1\}\cup\{(x,0)\in\mathbb{R}^2 : -1\le x\le 1\}$
The closure of A: $\operatorname{cl} A = A'$.

8. Calculate the following limits if they exist.

a) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\alpha x^2 y+\beta y^2}{x^4+y^2}$ **b)** $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^{2\beta}y+\sin(x^2+y^2)^{\alpha}}{(x^2+y^2)^{\beta}}$

Solution.

a) The limit at the origin along the parabolas $y=cx^2$ is $\displaystyle\lim_{x\to 0}\frac{\alpha c x^4+\beta c^2x^4}{x^4+c^2x^4}=\frac{\alpha c+\beta c^2}{1+c^2}$, which depends on c, that is, the limit depends on the route, therefore the limit at the origin doesn’t exist.

b) Using polar coordinates, $x=r\cos\varphi$, $y=r\sin\varphi$, where $r\longrightarrow 0$ and φ is arbitrary, the limit at the origin is $\displaystyle\lim_{r\to 0}\frac{r^{2\beta+1}(\sin\varphi)(\cos\varphi)^{2\beta}+\sin(r^{2\alpha})}{r^{2\beta}}=\lim_{r\to 0} r\cdot(\sin\varphi)(\cos\varphi)^{2\beta}+\lim_{r\to 0}\frac{\sin(r^{2\alpha})}{r^{2\beta}}=$

$$=0+\lim_{r\to 0}\frac{\sin(r^{2\alpha})}{r^{2\beta}}=\begin{cases}0, & \text{if } \alpha>\beta\\ 1, & \text{if } \alpha=\beta.\\ \infty, & \text{if } \alpha<\beta\end{cases}$$