Improper integrals

Case 1: The interval is not bounded

Definition. Let $a \in \mathbb{R}$ and assume that f is Riemann integrable on $[a, b]$ for all $b \geq a$.

If the limit $\lim\limits_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x \in \mathbb{R}$ exists then we say that f is **improperly integrable** or f has an **improper integral** on $[a, \infty)$ and the value of the integral is

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x.$$

In this case we also say that **the improper integral converges**.

If the limit $\lim\limits_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$ doesn't exist or if $\lim\limits_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x = \infty$ or $-\infty$ then we say that f is not improperly integrable on $[a, \infty)$ or **the improper integral diverges**.

Definition. Similarly, let $b \in \mathbb{R}$ and assume that f is Riemann integrable on $[a, b]$ for all $a \leq b$.

Then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^b f(x)\,\mathrm{d}x.$$

If the limit exists and is finite then the improper integral converges.

If the limit doesn't exist or exists but is ∞ or $-\infty$ then the improper integral diverges.

Examples

Exercise 1. Calculate the following integrals:

a) $\displaystyle\int_0^\infty \frac{1}{(1+x)^2}\,\mathrm{d}x$ b) $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ c) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

d) $\displaystyle\int_0^\infty \cos t\,\mathrm{d}t$ e) $\displaystyle\int_0^\pi \frac{1}{1+\sin x}\,\mathrm{d}x$ f) $\displaystyle\int_{-\infty}^a e^x\,\mathrm{d}x$

Results:

a) 1 b) ∞ (diverges) c) $\dfrac{\pi}{2}$ d) doesn't exist (diverges) e) 2 f) e^a

Exercise 2. $I = \displaystyle\int_2^\infty \frac{6}{x^2+x-2}\,\mathrm{d}x = ?$

Solution. Using partial fraction decomposition:

$$\frac{6}{x^2+x-2} = \frac{6}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \implies 6 = A(x+2) + B(x-1)$$

If $x = -2$: $B = -2$

If $x = 1$: $A = 2$

$$I = \lim_{b\to\infty} \int_2^b \frac{6}{x^2+x-2}\,\mathrm{d}x = \lim_{b\to\infty} \int_2^b \left(\frac{2}{x-1} - \frac{2}{x+2}\right)\mathrm{d}x =$$

$$= 2 \lim_{b \to \infty} [\ln(x-1) - \ln(x+2)]_2^b = 2 \lim_{b \to \infty} (\ln(b-1) - \ln(b+2) - (\ln 1 - \ln 4)) =$$

$$= 2 \lim_{b \to \infty} \left(\ln \frac{b-1}{b+2} + \ln 4 \right) = 2 \cdot (0 + \ln 4) = 2 \ln 4$$

(the improper integral converges)

Exercise 3. $I = \int_{-\infty}^{-1} \frac{1}{(x-2)\sqrt{\ln(2-x)}} \, dx = ?$

Solution. We use that $\int f' f^{\alpha} = \frac{f^{\alpha+1}}{\alpha+1} + c, \ \alpha \neq -1$:

$$I = \lim_{a \to -\infty} \int_a^{-1} \frac{-1}{2-x} (\ln(2-x))^{-\frac{1}{2}} dx =$$

$$= \lim_{a \to -\infty} \left[\frac{(\ln(2-x))^{\frac{1}{2}}}{\frac{1}{2}} \right]_a^{-1} = 2 \lim_{a \to -\infty} \left(\sqrt{\ln 3} - \sqrt{\ln(2-a)} \right) = -\infty$$

(the improper integral diverges)

Important remark

Definition. Let $a, b \in \overline{\mathbb{R}}$. The improper integral $I = \int_a^b f(x) \, dx$ is said to be convergent if for all $c \in (a, b)$ the improper integrals

$$I_1 = \int_a^c f(x) \, dx \ \text{ and } \ I_2 = \int_c^b f(x) \, dx$$

are both convergent.
The improper integral I is divergent if at least one of I_1 and I_2 is divergent.

Definition. $\int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f(x) \, dx$ if the double limit exists and is finite.

Remark. Because of the previous definition $\int_{-\infty}^{\infty} f(x) \, dx \neq \lim_{a \to \infty} \int_{-a}^{a} f(x) \, dx$.

For example, $\int_{-\infty}^{\infty} x \, dx$ is divergent, since $\int_{-\infty}^{0} x \, dx = -\infty$ and $\int_0^{\infty} x \, dx = \infty$.

However, $\lim_{a \to \infty} \int_{-a}^{a} x \, dx = \lim_{a \to \infty} \left[\frac{x^2}{2} \right]_{-a}^{a} = \lim_{a \to \infty} \left(\frac{a^2}{2} - \frac{a^2}{2} \right) = 0.$

Exercise 4. $I = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = ?$

Result: $I = \pi$

Case 2: The function is not bounded

Definition. Assume that f is not bounded at a but f is Riemann integrable on $[A, b]$ for all $a < A \le b$.
Then

$$\int_a^b f(x)\,dx = \lim_{\delta\to+0}\int_{a+\delta}^b f(x)\,dx \quad\text{or}\quad \int_a^b f(x)\,dx = \lim_{A\to a+0}\int_A^b f(x)\,dx$$

Definition. Assume that f is not bounded at b but f is Riemann integrable on $[a, B]$ for all $a \le B < b$.
Then

$$\int_a^b f(x)\,dx = \lim_{\delta\to+0}\int_a^{b-\delta} f(x)\,dx \quad\text{or}\quad \int_a^b f(x)\,dx = \lim_{B\to b-0}\int_a^B f(x)\,dx$$

Definition. If f is not bounded at $c \in (a, b)$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \lim_{\delta_1\to+0}\int_a^{c-\delta_1} f(x)\,dx + \lim_{\delta_2\to+0}\int_{c+\delta_2}^b f(x)\,dx$$

Examples

Exercise 5. Calculate the following integrals:

a) $\displaystyle\int_0^1 \sqrt{\frac{1}{1-t}}\,dt$ b) $\displaystyle\int_0^{\pi/2} \sqrt{1+\sin x}\,dx$ c) $\displaystyle\int_0^1 \ln x\,dx$ d) $\displaystyle\int_0^1 \frac{1}{1-x^2}\,dx$

Results:
a) 2 b) 2 c) −1 d) ∞ (diverges)

Exercise 6. $I = \displaystyle\int_5^7 \frac{1}{\sqrt[3]{(x-5)^2}}\,dx = ?$

Solution. $I = \displaystyle\lim_{\delta\to+0}\int_{5+\delta}^7 (x-5)^{-\frac{2}{3}}\,dx = \lim_{\delta\to+0}\left[\frac{(x-5)^{\frac{1}{3}}}{\frac{1}{3}}\right]_{5+\delta}^7 = 3\lim_{\delta\to+0}\left(\sqrt[3]{2} - \sqrt[3]{\delta}\right) = 3\sqrt[3]{2}.$

Exercise 7. $I = \displaystyle\int_0^1 \frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}}\,dx = ?$

Solution. We use that $\displaystyle\int f' f^\alpha = \frac{f^{\alpha+1}}{\alpha+1} + c,\ \ \alpha \ne -1$:

$$I = \lim_{\delta\to+0}\int_0^{1-\delta} \frac{1}{\sqrt{1-x^2}}(\arcsin x)^{\frac{1}{2}}\,dx = \lim_{\delta\to+0}\left[\frac{(\arcsin x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{1-\delta} = \frac{2}{3}\lim_{\delta\to+0}\left((\arcsin(1-\delta))^{\frac{3}{2}} - 0\right) = \frac{2}{3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}}$$

Improper integrals of $f(x) = \frac{1}{x^\alpha}$

Statement. The improper integral $\int_1^\infty \frac{1}{x^\alpha}\,dx$ is convergent if $\alpha > 1$ and divergent if $\alpha \le 1$.

Proof.

If $\alpha = 1$, then $\int_1^\infty \frac{1}{x^\alpha}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{x}\,dx = \lim_{A\to\infty} [\ln x]_1^A = \lim_{A\to\infty} (\ln A - \ln 1) = \infty$

If $\alpha \neq 1$, then $\int_1^\infty \frac{1}{x^\alpha}\,dx = \lim_{A\to\infty} \int_1^A x^{-\alpha}\,dx = \lim_{A\to\infty} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^A = \lim_{A\to\infty} \left(\frac{A^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}\right) = \begin{cases} 0 - \frac{1}{-\alpha+1} = \frac{1}{\alpha-1}, & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha < 1 \end{cases}$

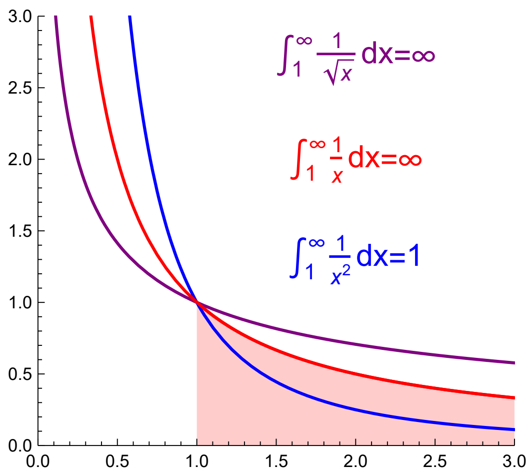
Statement. The improper integral $\int_0^1 \frac{1}{x^\alpha}\,dx$ is convergent if $\alpha < 1$ and divergent if $\alpha \ge 1$.

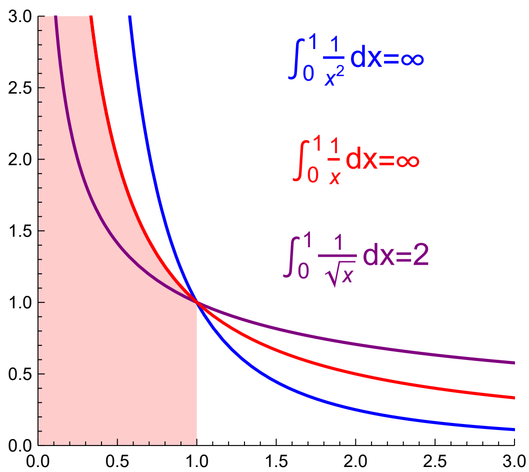
Proof.

If $\alpha = 1$, then $\int_0^1 \frac{1}{x^\alpha}\,dx = \lim_{\varepsilon\to 0+0} \int_{0+\varepsilon}^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0+0} [\ln x]_\varepsilon^1 = \lim_{\varepsilon\to 0+0} (\ln 1 - \ln \varepsilon) = 0 - (-\infty) = \infty$

If $\alpha \neq 1$, then $\int_0^1 \frac{1}{x^\alpha}\,dx = \lim_{\varepsilon\to 0+0} \int_{0+\varepsilon}^1 x^{-\alpha}\,dx = \lim_{\varepsilon\to 0+0} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0+0} \left(\frac{1}{-\alpha+1} - \frac{\varepsilon^{-\alpha+1}}{-\alpha+1}\right) =$

$$= \begin{cases} \infty, & \text{if } \alpha > 1 \\ \frac{1}{-\alpha+1} - 0 = \frac{1}{1-\alpha}, & \text{if } \alpha < 1 \end{cases}$$





Convergence of improper integrals

Theorem (Cauchy criterion for improper integrals).
Assume that f is integrable on closed, bounded subintervals of $[a, \beta)$.
(Here β can be ∞.)

Then the improper integral $\int_a^\beta f(x)\,dx$ is convergent if and only if

for all $\varepsilon > 0$ there exists $b \in [a, \beta)$ such that $\left| \int_{b_1}^{b_2} f(x)\,dx \right| < \varepsilon$ if $b < b_1 < b_2 < \beta$.

Remark. Assume that f is integrable on closed, bounded subintervals of $[\alpha, \beta)$.

If $f \geq 0$ on $[\alpha, \beta)$ then $g(\omega) = \int_\alpha^\omega f(x)\,dx$ is monotonically increasing

$\Longrightarrow \lim\limits_{\omega \to \beta - 0} \int_\alpha^\omega f(x)\,dx$ exists (it is either finite or ∞)

$\Longrightarrow \int_\alpha^\beta |f(x)|\,dx$ always exists if f is integrable on closed, bounded subintervals of $[\alpha, \beta)$.

Definition. The improper integral $\int_\alpha^\beta f(x)\,dx$ is **absolutely convergent**, if

f is integrable on closed, bounded subintervals of $[\alpha, \beta)$ and

$\int_\alpha^\beta |f(x)|\,dx$ is convergent.

Theorem. If the improper integral $\int_\alpha^\beta f(x)\,dx$ is absolutely convergent then

the improper integral $\int_\alpha^\beta f(x)\,dx$ is convergent.

Proof. $\left| \int_{b_1}^{b_2} f(x)\,dx \right| \leq \int_{b_1}^{b_2} |f(x)|\,dx$ for all $\alpha \leq b_1 < b_2 < \beta$

and we use the Cauchy criterion.

Remark. The converse of the statement is not necessarily true, for example

$\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent, but not absolutely convergent (see later).

Comparison test for improper integrals

Theorem. Assume that

- f and g are integrable on closed, bounded subintervals of $[\alpha, \beta)$
- $\exists\, b_0 \in [\alpha, \beta)$ such that $|f(x)| \leq g(x)\ \forall x \in (b_0, \beta)$

Then

1. if $\int_\alpha^\beta g(x)\,dx$ converges then $\int_\alpha^\beta f(x)\,dx$ also converges.
2. if $g(x) \geq |f(x)|\ \forall x \in (b_0, \beta)$ and $\int_\alpha^\beta f(x)\,dx$ diverges then $\int_\alpha^\beta g(x)\,dx$ also diverges.

Proof. 1. • $\int_\alpha^\beta g(x)\,dx$ converges $\overset{\text{Cauchy}}{\Longleftrightarrow}$ $\forall \varepsilon > 0\ \exists\, b < \beta$ such that $\left| \int_{b_1}^{b_2} g(x)\,dx \right| < \varepsilon$ for all $b < b_1 < b_2 < \beta$.

• $|f(x)| \leq g(x)\ \forall x \in (b_0, \beta)$

$\Longrightarrow \left| \int_{b_1}^{b_2} f(x)\,dx \right| \leq \left| \int_{b_1}^{b_2} |f(x)|\,dx \right| \leq \left| \int_{b_1}^{b_2} g(x)\,dx \right| < \varepsilon$ for all $\max\{b, b_0\} < b_1 < b_2 < \beta$

$\overset{\text{Cauchy}}{\Longrightarrow} \int_\alpha^\beta f(x)\,dx$ converges.

2. By part 1, if $\int_\alpha^\beta g(x)\,dx$ is convergent then $\int_\alpha^\beta f(x)\,dx$ is convergent, which is a contradiction.

Remark. 1. If $x \geq 1$ then $\frac{1}{x} \leq \frac{1}{\sqrt{x}}$. Since $\int_1^\infty \frac{1}{x}\,\mathrm{dx}$ diverges then $\int_1^\infty \frac{1}{\sqrt{x}}\,\mathrm{dx}$ also diverges.

2. If $0 < x \leq 1$ then $\frac{1}{x} \leq \frac{1}{x^2}$. Since $\int_0^1 \frac{1}{x}\,\mathrm{dx}$ diverges then $\int_0^1 \frac{1}{x^2}\,\mathrm{dx}$ also diverges.

Examples

Exercise 8. Decide whether the following integrals converge or diverge:

a) $\int_1^\infty \frac{\sin x}{x^2}\,\mathrm{dx}$ b) $\int_0^\infty \frac{1+\sin x}{1+x^2}\,\mathrm{dx}$ c) $\int_1^\infty \frac{2+\cos x}{x}\,\mathrm{dx}$

d) $\int_0^1 \frac{\sin x}{x^2}\,\mathrm{dx}$ e) $\int_1^\infty \frac{\sin x}{x}\,\mathrm{dx}$ f) $\int_1^\infty \left|\frac{\sin x}{x}\right|\,\mathrm{dx}$

Results:

a) conv. b) conv. c) div. d) div. e) conv. f) div.

Integral test

Theorem. Assume that $f : [1, \infty) \longrightarrow \mathbb{R}$ be a positive valued, monotonically decreasing function and let $f(k) = a_k > 0$.

1. If $\int_1^\infty f(x)\,\mathrm{dx}$ is convergent $\Longrightarrow$ $\sum_{k=1}^{\infty} a_k$ is convergent

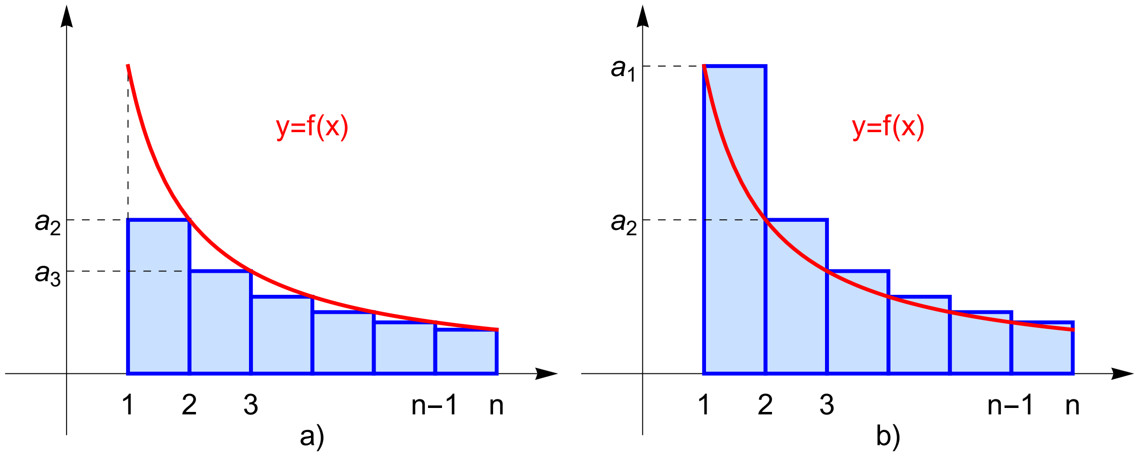
2. If $\int_1^\infty f(x)\,\mathrm{dx}$ is divergent $\Longrightarrow$ $\sum_{k=1}^{\infty} a_k$ is divergent

Remark. The equivalence is also true, that is, the integral $\int_1^\infty f(x)\,\mathrm{dx}$ and the series $\sum_{k=1}^{\infty} a_k$ are both convergent or both divergent.

Proof. 1. Consider Figure a). Since the sum of the areas of the inscribed rectangles is less than or equal to the area under the graph of f then

$$a_2 + a_3 + \ldots + a_n \leq \int_1^n f(x)\,\mathrm{dx} \leq \lim_{n\to\infty} \int_1^n f(x)\,\mathrm{dx} = \int_1^\infty f(x)\,\mathrm{dx} \in \mathbb{R}.$$

Since $a_k > 0$ and $\sum_{k=2}^{n} a_k$ is bounded $\Longrightarrow$ $\sum_{k=2}^{\infty} a_k$ is convergent $\Longrightarrow$ $\sum_{k=1}^{\infty} a_k$ is convergent.



2. Consider Figure b). Since the sum of the areas of the circumscribed rectangles is greater than or equal to the area under the graph of f then

$$\int_1^n f(x)\,\mathrm{dx} \le a_1 + a_2 + \ldots + a_{n-1} = s_{n-1}$$

Since $\lim_{n\to\infty} \int_1^n f(x)\,\mathrm{dx} = \infty \implies \lim_{n\to\infty} s_{n-1} = \infty \implies \sum_{k=1}^{\infty} a_k$ is divergent.

Error estimation

Theorem: Let $f : [1, \infty) \longrightarrow \mathbb{R}$ be a positive valued, monotonically decreasing function, let $f(k) = a_k > 0$ and suppose that $\int_1^\infty f(x)\,\mathrm{dx}$ is convergent. Let $s_n = \sum_{k=1}^{n} a_k$ and $s = \sum_{k=1}^{\infty} a_k$.

Then the error for the approximation $s \approx s_n$ is

$0 < E = s - s_n = \sum_{k=n+1}^{\infty} a_k \le \int_n^\infty f(x)\,\mathrm{dx}.$

Proof: Since $a_{n+1} + a_{n+2} + \ldots + a_m \le \int_n^m f(x)\,\mathrm{dx}$ then

$$0 < E = s - s_n = \lim_{m\to\infty} \sum_{k=n+1}^{m} a_k \le \lim_{m\to\infty} \int_n^m f(x)\,\mathrm{dx} = \int_n^\infty f(x)\,\mathrm{dx}.$$

The convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$

Theorem: $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent if $\alpha > 1$ and divergent otherwise.

Proof: If $\alpha < 0$ then $\lim_{n\to\infty} \frac{1}{n^\alpha} = \lim_{n\to\infty} n^{-\alpha} = \lim_{n\to\infty} n^{|\alpha|} = \infty$ and

if $\alpha = 0$ then $\lim_{n\to\infty} \frac{1}{n^\alpha} = 1$, so in these cases the series is divergent by the nth term test.

If $\alpha > 0$ then let $f(x) = \frac{1}{x^\alpha}$, $x \ge 1$. This function is positive valued, monotonic decreasing and $f(n) = \dfrac{1}{n^\alpha} = a_n > 0$.

Thus, we can apply the integral test to investigate the convergence of $\sum\limits_{n=1}^{\infty} a_n$.

We already proved that $\int_1^\infty \frac{1}{x^\alpha}\,dx$ is convergent if $\alpha > 1$, therefore the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^\alpha}$ is also convergent if $\alpha > 1$.

The improper integral is divergent if $0 < \alpha \leq 1$, so in this case the series is also divergent.

Examples

Exercise. Decide whether the following series are convergent or divergent.

a) $\sum\limits_{n=3}^{\infty} \dfrac{1}{n \ln n^{10}}$ b) $\sum\limits_{n=3}^{\infty} \dfrac{1}{n\left(\ln \sqrt{n}\right)^2}$

If a series is convergent then estimate the error for the approximation $s \approx s_{1000}$

Solution.

a) Let $f(x) = \dfrac{1}{x \ln x^{10}} = \dfrac{1}{10} \dfrac{1}{x \ln x}$, $x > 3$

Then f is positive valued and monotonically decreasing on the interval $[3, \infty)$ and $a_n = f(n) > 0$ $\Longrightarrow$ the integral test can be applied:

$$\int_3^\infty \frac{1}{10} \frac{1}{x \ln x}\,dx = \frac{1}{10} \lim_{A\to\infty} \int_3^A \frac{\frac{1}{x}}{\ln x}\,dx = \frac{1}{10} \lim_{A\to\infty} [\ln(\ln x)]_3^A = \frac{1}{10} \lim_{A\to\infty} (\ln(\ln A) - \ln(\ln 3)) = \infty$$

Since the improper integral is divergent then the series $\sum\limits_{n=3}^{\infty} \dfrac{1}{n \ln n^{10}}$ is also divergent by the integral test.

b) Let $f(x) = \dfrac{1}{x\left(\ln \sqrt{x}\right)^2} = \dfrac{1}{\frac{1}{4}} \cdot \dfrac{1}{x (\ln x)^2}$, $x > 3$

Then f is positive valued and monotonically decreasing on the interval $[3, \infty)$ and $a_n = f(n) > 0$ $\Longrightarrow$ the integral test can be applied:

$$\int_3^\infty 4 \cdot \frac{1}{x (\ln x)^2}\,dx = 4 \lim_{A\to\infty} \int_3^A \frac{1}{x} (\ln x)^{-2}\,dx = 4 \lim_{A\to\infty} \left[-\frac{1}{\ln x}\right]_3^A = 4 \lim_{A\to\infty} \left(-\frac{1}{\ln A} + \frac{1}{\ln 3}\right) = 4\left(0 + \frac{1}{\ln 3}\right) = \frac{4}{\ln 3}$$

Since the improper integral is convergent then the series $\sum\limits_{n=3}^{\infty} \dfrac{1}{n\left(\ln \sqrt{n}\right)^2}$ is also convergent by the integral test.

Error estimation for the approximation $s \approx s_{1000}$:

$$0 < E = s - s_{1000} \leq \int_{1000}^\infty f(x)\,dx = \int_{1000}^\infty 4 \cdot \frac{1}{x} \cdot (\ln x)^{-2}\,dx = 4 \lim_{A\to\infty} \left[-\frac{1}{\ln x}\right]_{1000}^A =$$

$$= 4 \lim_{A\to\infty} \left(-\frac{1}{\ln A} + \frac{1}{\ln 1000}\right) = \frac{4}{\ln 1000}.$$