Practice exercises 5.

1. The equation $z = x^2 y + x y^2 + x + 3y - 1$ defines a landscape, and at the point (4, 1, 26) of this landscape there is a spring. In which direction will the water flow from the spring?

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^3$, $f(t) = \left(t^2 - t, \frac{1}{1+t^2}, e^t\right)$ and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $g(x, y, z) = x^2 y - z$.

a) Calculate the derivative of $g \circ f$ at $t_0 = 1$ using the chain rule.

b) Calculate the derivative of $f \circ g$ at $a_0 = (2, 3, 11)$ using the chain rule.

3. Show that the following functions satisfy the the given differential equations:

a) $z(x, y) = e^{-a y} \cos a x, \quad \frac{\partial^2 z}{\partial x^2} = a \frac{\partial z}{\partial y}$

b) $u(x, t) = \sin(x - a t) + \ln(x + a t), \qquad u_{tt} = a^2 u_{xx}$

c) $u(x, y) = \sin x \cosh y + \cos x \sinh y, \qquad u_{xx} + u_{yy} = 0$

4. Show that for the function $f(x, y) = \begin{cases} \frac{x^4 + x y^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$ we obtain $f_{xy} \neq f_{yx}$.

5. Give the nth order Taylor polynomial $T_n(x, y)$ of the following functions at the point $P_0(x_0, y_0)$:

a) $f(x, y) = 2x^2 - x y - y^2 - 6x - 3y + 5$, $P_0(1, -2)$, $T_2(x, y) = ?$

b) $f(x, y, z) = x^3 + y^3 + z^3$, $P_0(1, 2, 3)$, $T_2(x, y) = ?$

c) $f(x, y) = \sin(x + 2y)$, $P_0\left(\frac{\pi}{4}, \frac{\pi}{6}\right)$, $T_2(x, y) = ?$

d) $f(x, y) = \frac{x}{y}$, $P_0(1, 1)$, $T_3(x, y) = ?$