Practice exercises 4.

1. Calculate the partial derivatives of the following functions:

a) $f:\mathbb{R}^3 \longrightarrow \mathbb{R}$, $f(x, y, z) = \dfrac{x^2 + \cos y}{1 + e^z} + y^2 z$

b) $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x, y) = \begin{cases} \exp\left(-\dfrac{1}{x^2 + y^2}\right), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

2. Show that the following function is not continuous at the origin, however, the partial derivatives exist at the origin:

$$f(x, y) = \begin{cases} \dfrac{x y}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

3. Calculate the partial derivatives of the following functions. Where are these functions differentiable?

a) $f(x, y) = \sqrt{5(x-1)^4 + 4y^2}$

b) $f(x, y) = \begin{cases} \dfrac{(x-2) y^2}{x^2 + y^2} + 6x + 3y, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

c) $f(x, y) = \begin{cases} \dfrac{\sin(y^2 + 2x^2)}{\sqrt{y^2 + 2x^2}}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

4. Find the equation of the tangent plane at the point P_0, and give the directional derivative of f at P_0 in the direction of $\boldsymbol{v}$.

a) $f(x, y) = 3y + e^{x y^2} - 2y \arctan \dfrac{x}{y}$, $P_0(0, 1)$, $\boldsymbol{v} = (2, 1)$

b) $f(x, y) = \begin{cases} \dfrac{x^2 - 3y^2}{2x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ -3, & \text{if } (x, y) = (0, 0) \end{cases}$, $P_0(-1, 1)$, $\boldsymbol{v} = (-5, 1)$

c) $f(x, y) = \dfrac{y^3}{e^{2x+1}}$, $P_0\left(-\dfrac{1}{2}, 1\right)$. Find the directional derivative of f at P_0 with the maximal and minimal value and also the corresponding direction.

5. Consider the surface given by the equation $x y z = 1$. Find the tangent plane parallel to the plane with equation $x + y + z = 3$.

6. Consider the surface given by the equation $x y z = a^3$ $(a > 0)$. Prove that the tangent planes of the surface form tetrahedra with constant volume with the coordinate planes.