Practice exercises 2.

1. Prove that

a) $\|x-y\| \geq |\ \|x\|-\|y\|\ |$ for all $x, y \in \mathbb{R}^p$;

b) $\|x-y\|^2+\|x+y\|^2 = 2\|x\|^2+2\|y\|^2$ for all $x, y \in \mathbb{R}^p$;

c) if $\|x\| = \|y\| = 1$ and $x \perp y$ (i.e. $< x, y > = 0$), then $\|x-y\| = \sqrt{2}$;

d) $(x-y) \perp (x+y)$ if and only if $\|x\| = \|y\|$.

2. Sketch the following subsets of $\mathbb{R}^2$, find the set of interior points, boundary points, limit points and isolated points and the closure of the sets.

a) $\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, x+y < 1\}$

b) $\{(x, 0) \in \mathbb{R}^2 : 0 < x < 1\}$

c) $\left\{(x, y) \in \mathbb{R}^2 : x = \frac{1}{n}\ (n = 1, 2, \ldots), 0 < y < 1\right\}$

d) $\left\{\left(-\frac{1}{n}, -\frac{1}{n}\right) \in \mathbb{R}^2 : n \in \mathbb{N}^+\right\} \cup]3, 4] \times \{0\}$

e) $\{(x, y) \in \mathbb{R}^2 : 0 < x, 0 < y < x^2\}$

f) $\left\{(x, y) \in \mathbb{R}^2 : 0 < x \leq 1, 0 \leq y \leq \sqrt{x}\right\} \cap [-1, 0[\times \{0\}$

g) $\left\{(x, y) \in \mathbb{R}^2 : 0 < x < 1, y = \sin\left(\frac{1}{x}\right)\right\}$

h) $\left\{(x, y) \in \mathbb{R}^2 : 0 < x, y < \sin\left(\frac{1}{x}\right)\right\}$

3. Consider $\mathbb{Q} \subset \mathbb{R}$. Find int($\mathbb{Q}$), $\partial\mathbb{Q}$, ext($\mathbb{Q}$).

4. Prove that if $A \neq \emptyset, \mathbb{R}^p$, then A cannot be open and closed at the same time.

5. Is there a set $A \subset \mathbb{R}^2$ such that $\partial A = \left\{\left(\frac{1}{n}, 0\right) : n = 1, 2, \ldots\right\}$?

6. a) Is the set $]1, 2[$ open in $\mathbb{R}$?

b) Is the set $]1, 2[\times \{0\}$ open in $\mathbb{R}^2$?

c) Is the set $[1, \infty[$ closed in $\mathbb{R}$?

d) Is the set $[1, \infty[\times \{0\}$ closed in $\mathbb{R}^2$?

Homework: see also the Quiz questions about Basic topological concepts in Calculus 1.