Some two-variable functions and their graphs

1. Planes

• $a\,x + b\,y + c\,z = d$ is the equation of a plane with normal vector (a, b, c)

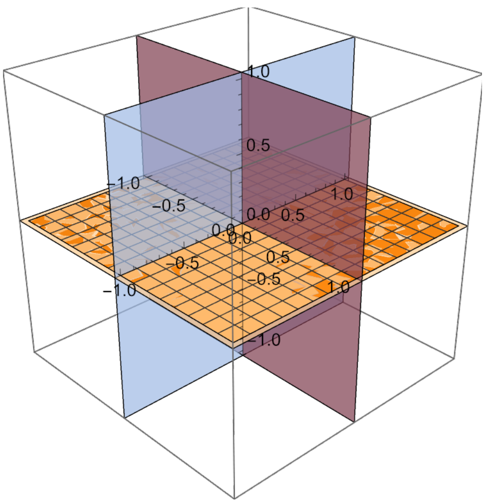
Some examples:

• $z = 0$ is the equation of the xy-plane (normal vector: $(0, 0, 1)$)

The graph of the constant function $f(x, y) = 0$ is the same.

• $x = 0$ is the equation of the yz-plane (normal vector: $(1, 0, 0)$)

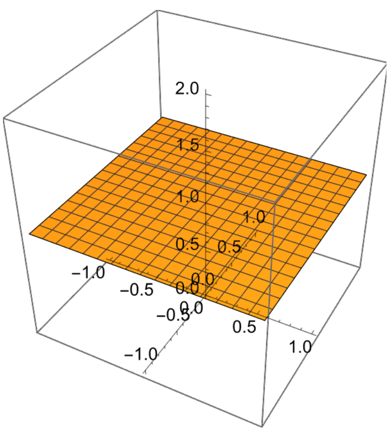
• $y = 0$ is the equation of the xz-plane (normal vector: $(0, 1, 0)$)



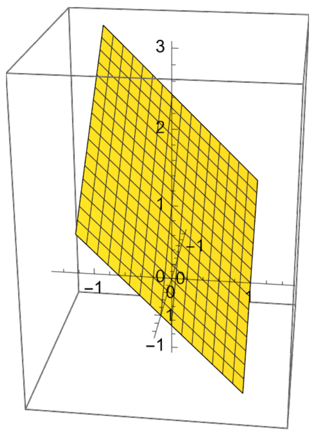
• $z = c$ is the equation of a horizontal plane, that is, parallel to the xy-plane, intersecting the z axis at $z = c$
(normal vector: $(0, 0, 1)$). The graph of the constant function $f(x, y) = c$ is the same.

• $x + y + z = 1 \implies z = 1 - x - y$ or $f(x, y) = 1 - x - y$ is a plane with normal vector $(1, 1, 1)$, intersecting the axes at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$

$z = 1$



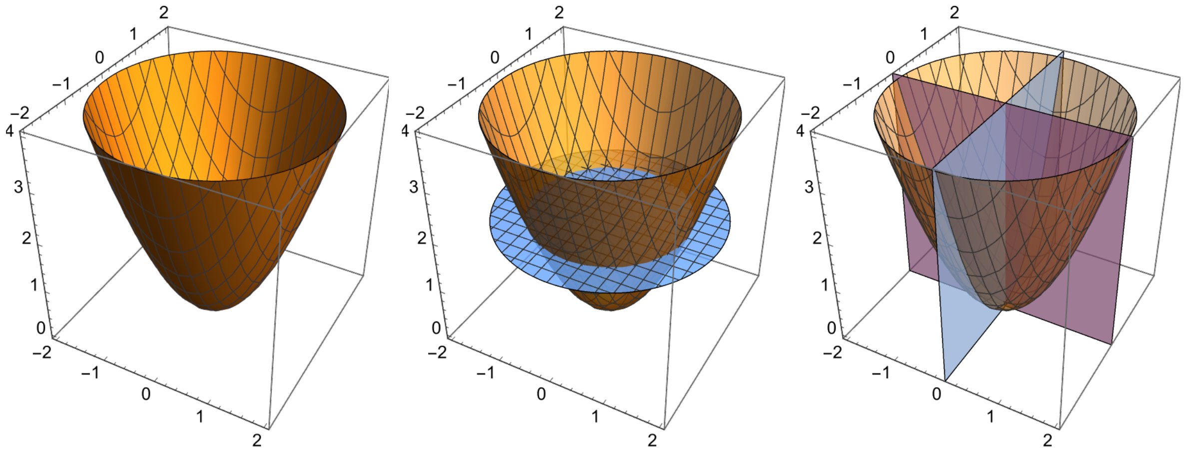
$z = 1 - x - y$



2. Paraboloid: $z = x^2 + y^2$ or $f(x, y) = x^2 + y^2$

Let us intersect the graph of *f* with horizontal and vertical planes. For example:

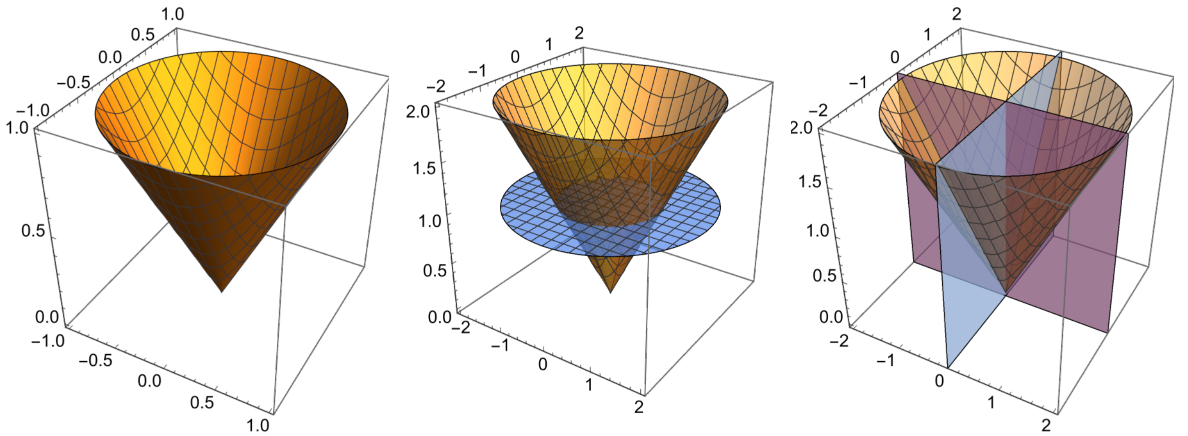
- The intersection of $z = x^2 + y^2$ and $\begin{cases} z = 1 \\ z = 4 \end{cases}$ is a horizontal circle at height $\begin{cases} z = 1 \\ z = 4 \end{cases}$ with radius $\begin{cases} r = 1 \\ r = 2 \end{cases}$
- The intersection of $z = x^2 + y^2$ and $\begin{cases} x = 0 \\ y = 0 \end{cases}$ is $\begin{cases} z = y^2 \\ z = x^2 \end{cases}$, this is a parabola in the $\begin{cases} yz - \text{plane} \\ xz - \text{plane} \end{cases}$



3. Cone: $z = \sqrt{x^2 + y^2}$ or $f(x, y) = \sqrt{x^2 + y^2}$

Let us intersect the graph of *f* with horizontal and vertical planes. For example:

- The intersection of $z = \sqrt{x^2 + y^2}$ and $\begin{cases} z = 1 \\ z = 2 \end{cases}$ is a horizontal circle at height $\begin{cases} z = 1 \\ z = 2 \end{cases}$ with radius $\begin{cases} r = 1 \\ r = 2 \end{cases}$
- The intersection of $z = \sqrt{x^2 + y^2}$ and $\begin{cases} x = 0 \\ y = 0 \end{cases}$ is $\begin{cases} z = |y| \\ z = |x| \end{cases}$, this is an absolute value function in the $\begin{cases} yz - \text{plane} \\ xz - \text{plane} \end{cases}$



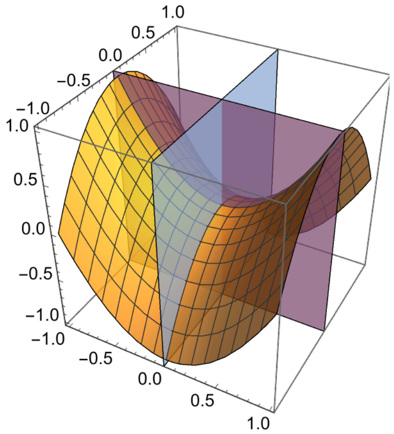
4. Saddle surfaces

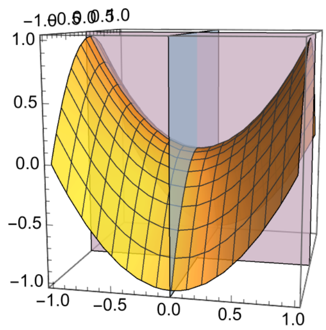
a) $z = x^2 - y^2$ or $f(x, y) = x^2 - y^2$

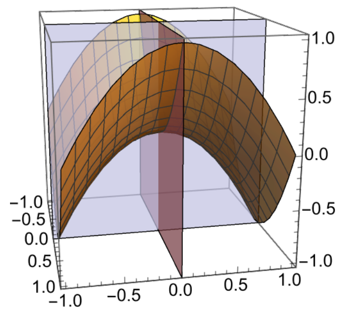
Let us intersect the graph of f with the xz-plane ($y = 0$) and with the yz-plane ($x = 0$).

- The intersection of $z = x^2 - y^2$ and $\begin{cases} x = 0 \\ y = 0 \end{cases}$ is $\begin{cases} z = -y^2 \\ z = x^2 \end{cases}$, this is a(n)

$\begin{cases} \text{downward parabola in the } yz - \text{plane} \\ \text{upward parabola in the } xz - \text{plane} \end{cases}$





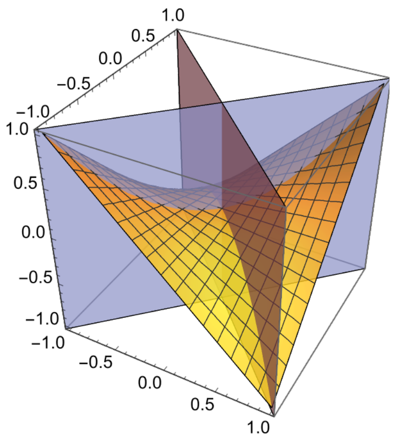


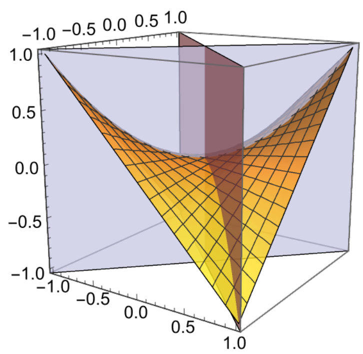
b) $z = x\,y$ or $f(x, y) = x\,y$

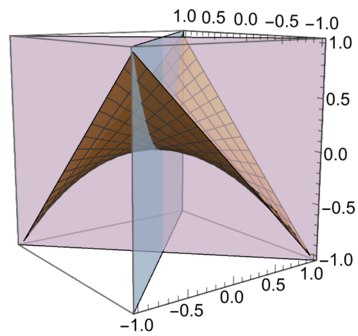
Let us intersect the graph of f with vertical planes that intersect the xy-plane in the straight lines $y = x$ and $y = -x$.

- The intersection of $z = x\,y$ and the vertical plane $\begin{cases} y = x \\ y = -x \end{cases}$ is $\begin{cases} z = x^2 \\ z = -x^2 \end{cases}$, this is a(n)

$\begin{cases} \text{upward parabola} \\ \text{downward parabola} \end{cases}$



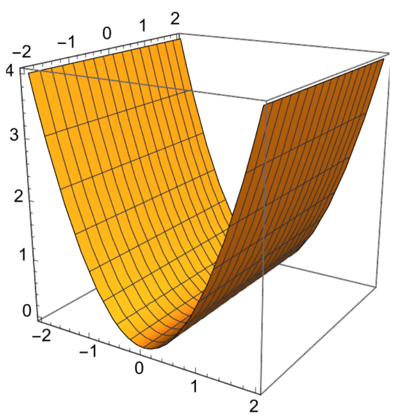




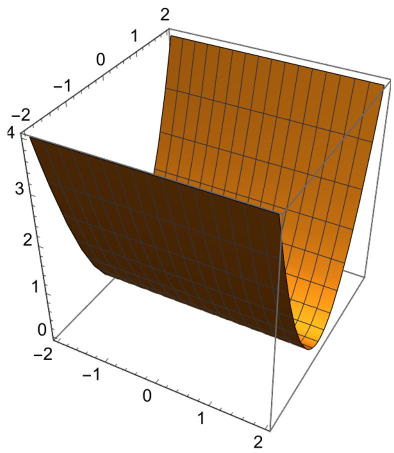
5. $z = x^2$ and $z = y^2$

Intersecting the surface $z = x^2$ with planes parallel to the xz-plane ($y = 0$), the intersection curves are parabolas.

$z = x^2$



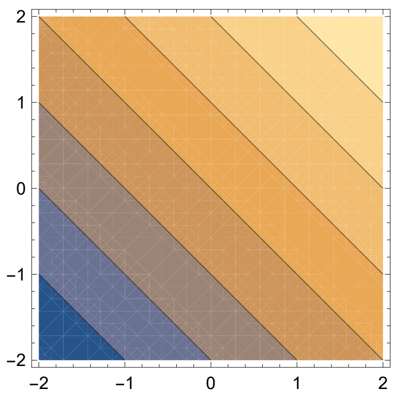
$z = y^2$



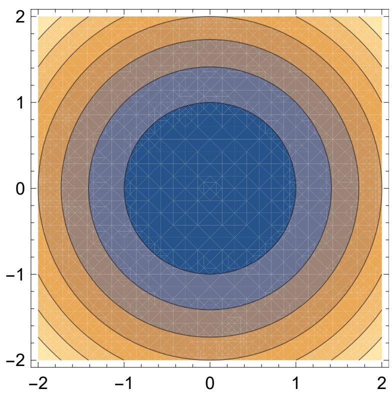
Contour lines

The surfaces can also be represented in two dimensions with contour lines. The surface is intersected with horizontal planes and the intersection curves are projected perpendicularly to the xy-plane. The contour lines are the projections of the intersection curves, that is, they denote the points of the same height, similarly to the contour lines on a map.

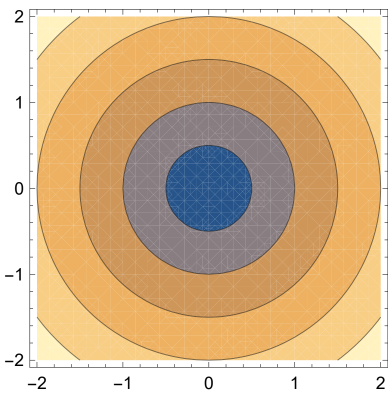
$z = x + y$ (plane)



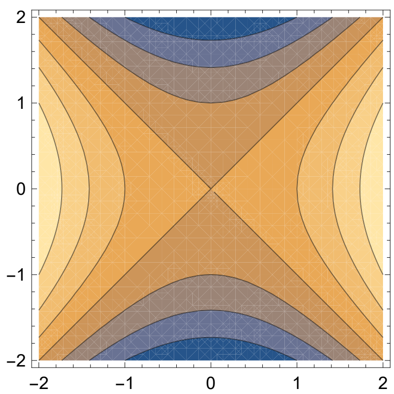
$z = x^2 + y^2$ (paraboloid)



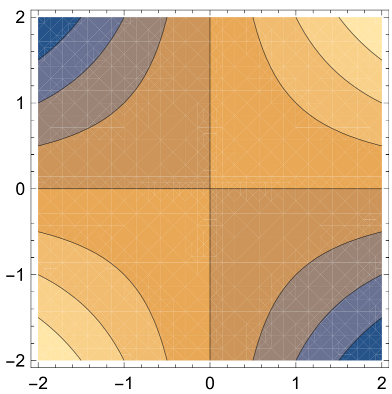
$z = \sqrt{x^2 + y^2}$ (cone)



$z = x^2 - y^2$ (saddle)



$z = x\,y$ (saddle)



$z = x^2$