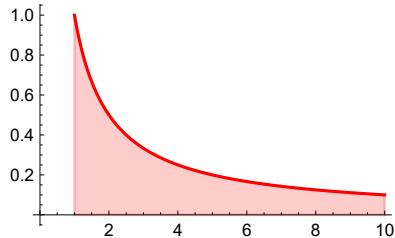


## Improper integrals, exercises

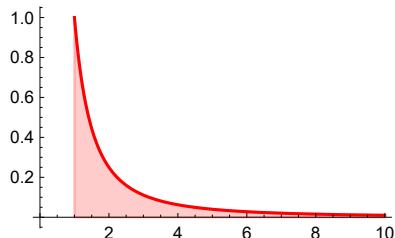
**Exercise 1.**  $\int_1^\infty \frac{1}{x} dx = ?$

**Solution:**  $\int_1^\infty \frac{1}{x} dx = \lim_{A \rightarrow \infty} \int_1^A \frac{1}{x} dx = \lim_{A \rightarrow \infty} [\ln x]_1^A = \lim_{A \rightarrow \infty} (\ln A - \ln 1) = \infty$



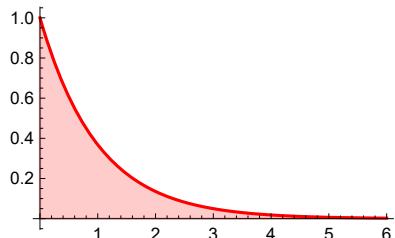
**Exercise 2.**  $\int_1^\infty \frac{1}{x^2} dx = ?$

**Solution:**  $\int_1^\infty \frac{1}{x^2} dx = \lim_{A \rightarrow \infty} \int_1^A \frac{1}{x^2} dx = \lim_{A \rightarrow \infty} \int_1^A x^{-2} dx = \lim_{A \rightarrow \infty} \left[ \frac{x^{-1}}{-1} \right]_1^A = \lim_{A \rightarrow \infty} \left[ -\frac{1}{x} \right]_1^A = \lim_{A \rightarrow \infty} \left( -\frac{1}{A} + 1 \right) = 0 + 1 = 1$



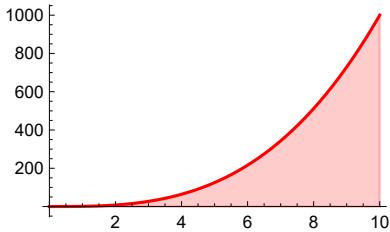
**Exercise 3.**  $\int_0^\infty e^{-x} dx = ?$

**Solution:**  $\int_0^\infty e^{-x} dx = \lim_{A \rightarrow \infty} \int_0^A e^{-x} dx = \lim_{A \rightarrow \infty} \left[ \frac{e^{-x}}{-1} \right]_0^A = \lim_{A \rightarrow \infty} [-e^{-x}]_0^A = \lim_{A \rightarrow \infty} (-e^{-A} - (-e^0)) = 0 + 1 = 1$



**Exercise 4.**  $\int_0^\infty x^3 dx = ?$

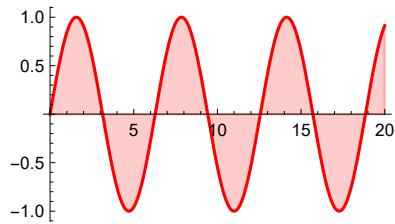
**Solution:**  $\int_0^\infty x^3 dx = \lim_{A \rightarrow \infty} \int_0^A x^3 dx = \lim_{A \rightarrow \infty} \left[ \frac{x^4}{4} \right]_0^A = \lim_{A \rightarrow \infty} \left( \frac{A^4}{4} - 0 \right) = \infty$



**Exercise 5.**  $\int_0^\infty \sin x dx = ?$

**Solution:**  $\int_0^\infty \sin x dx = \lim_{A \rightarrow \infty} \int_0^A \sin x dx = \lim_{A \rightarrow \infty} [-\cos x]_0^A = \lim_{A \rightarrow \infty} (-\cos A + \cos 0),$

this limit doesn't exist, so the integral is divergent.

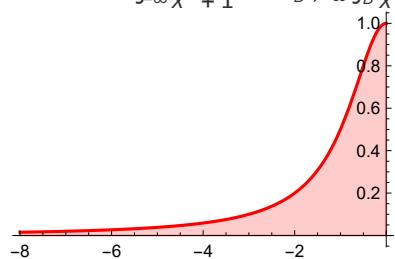


**Exercise 6.**  $\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}} dx = ?$

**Solution:**  $\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}} dx = \lim_{B \rightarrow -\infty} \int_B^{-1} \frac{1}{\sqrt[3]{x}} dx = \lim_{B \rightarrow -\infty} \int_B^{-1} x^{-\frac{1}{3}} dx = \lim_{B \rightarrow -\infty} \left[ \frac{x^{\frac{2}{3}}}{\frac{2}{3}} \right]_B^{-1} =$   
 $= \lim_{B \rightarrow -\infty} \left[ \frac{3}{2} \cdot \sqrt[3]{x^2} \right]_B^{-1} = \lim_{B \rightarrow -\infty} \left( \frac{3}{2} \cdot \sqrt[3]{(-1)^2} - \frac{3}{2} \cdot \sqrt[3]{B^2} \right) = \frac{3}{2} - \infty = -\infty$

**Exercise 7.**  $\int_{-\infty}^0 \frac{1}{x^2 + 1} dx = ?$

**Solution:**  $\int_{-\infty}^0 \frac{1}{x^2 + 1} dx = \lim_{B \rightarrow -\infty} \int_B^0 \frac{1}{x^2 + 1} dx = \lim_{B \rightarrow -\infty} [\arctg x]_B^0 = \lim_{B \rightarrow -\infty} (\arctg 0 - \arctg B) = 0 - \left( -\frac{\pi}{2} \right) = \frac{\pi}{2}$

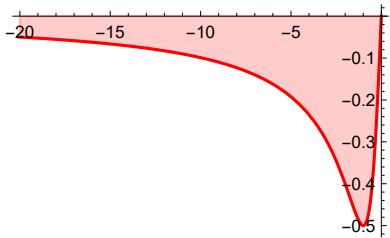


**Exercise 8.**  $\int_{-\infty}^0 \frac{x}{x^2 + 1} dx = ?$

**Solution:**  $\int_{-\infty}^0 \frac{x}{x^2 + 1} dx = \lim_{B \rightarrow -\infty} \int_B^0 \frac{x}{x^2 + 1} dx = \lim_{B \rightarrow -\infty} \int_B^0 \frac{1}{2} \cdot \frac{2x}{x^2 + 1} dx =$

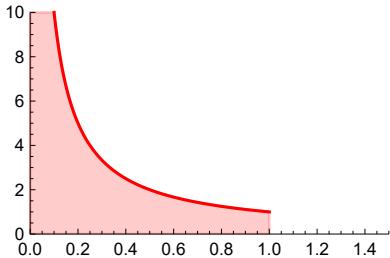
(the integrand has the form  $\frac{f'}{f}$ )

$$= \lim_{B \rightarrow -\infty} \left[ \frac{1}{2} \ln(x^2 + 1) \right]_B^0 = \lim_{B \rightarrow -\infty} \left( \frac{1}{2} \ln 1 - \frac{1}{2} \ln(B^2 + 1) \right) = 0 - \infty = -\infty$$



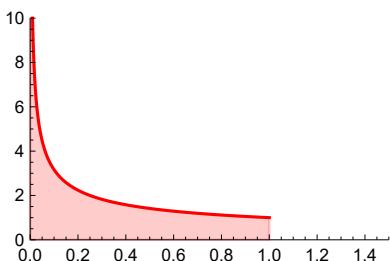
**Exercise 9.**  $\int_0^1 \frac{1}{x} dx = ?$

**Solution:**  $\int_0^1 \frac{1}{x} dx = \lim_{\varepsilon \rightarrow 0+0} \int_{0+\varepsilon}^1 \frac{1}{x} dx = \lim_{\varepsilon \rightarrow 0+0} [\ln x]_0^1 = \lim_{\varepsilon \rightarrow 0+0} (\ln 1 - \ln \varepsilon) = 0 - (-\infty) = \infty$



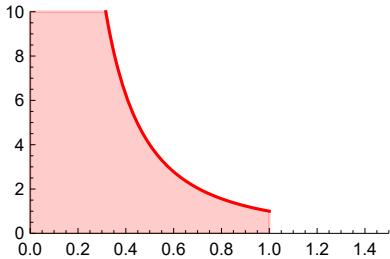
**Exercise 10.**  $\int_0^1 \frac{1}{\sqrt{x}} dx = ?$

**Solution:**  $\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{\varepsilon \rightarrow 0+0} \int_{0+\varepsilon}^1 \frac{1}{\sqrt{x}} dx = \lim_{\varepsilon \rightarrow 0+0} \int_{\varepsilon}^1 x^{-\frac{1}{2}} dx = \lim_{\varepsilon \rightarrow 0+0} \left[ \frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^1 =$   
 $= \lim_{\varepsilon \rightarrow 0+0} [2 \sqrt{x}]_0^1 = \lim_{\varepsilon \rightarrow 0+0} (2 \sqrt{1} - 2 \sqrt{\varepsilon}) = 2 - 0 = 2$



**Exercise 11.**  $\int_0^1 \frac{1}{x^2} dx = ?$

**Solution:**  $\int_0^1 \frac{1}{x^2} dx = \lim_{\varepsilon \rightarrow 0+0} \int_{0+\varepsilon}^1 \frac{1}{x^2} dx = \lim_{\varepsilon \rightarrow 0+0} \int_{\varepsilon}^1 x^{-2} dx = \lim_{\varepsilon \rightarrow 0+0} \left[ \frac{x^{-1}}{-1} \right]_0^1 =$   
 $= \lim_{\varepsilon \rightarrow 0+0} \left[ -\frac{1}{x} \right]_0^1 = \lim_{\varepsilon \rightarrow 0+0} \left( -1 + \frac{1}{\varepsilon} \right) = -1 + \frac{1}{0+} = -1 + \infty = \infty$



**Exercise 12.**  $\int_0^1 \ln x dx = ?$

**Solution:** By integration by parts:  $\int 1 \cdot \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + c$

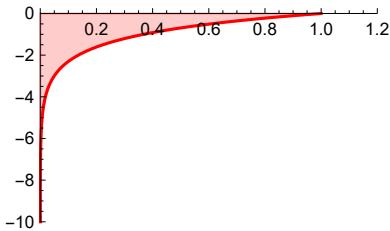
$$f'(x) = 1 \implies f(x) = x$$

$$g(x) = \ln x \implies g'(x) = \frac{1}{x}$$

$$\begin{aligned} \int_0^1 \ln x dx &= \lim_{\epsilon \rightarrow 0+0} \int_{0+\epsilon}^1 \ln x dx = \lim_{\epsilon \rightarrow 0+0} [x \ln x - x]_{\epsilon}^1 = \\ &= \lim_{\epsilon \rightarrow 0+0} ((1 \cdot \ln 1 - 1) - (\epsilon \ln \epsilon - \epsilon)) = (0 - 1) - (0 - 0) = -1 \end{aligned}$$

**By L'Hospital's rule** (the product has the form  $0 \cdot (-\infty)$ ):

$$\lim_{x \rightarrow 0+0} x \ln x = \lim_{x \rightarrow 0+0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0+0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0+0} (-x) = 0.$$



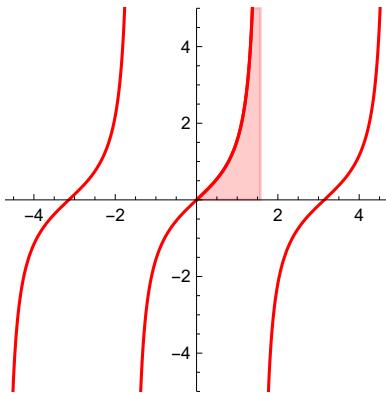
**Exercise 13.**  $\int_0^{\pi/2} \tan x dx = ?$

**Solution:**  $\int_0^{\pi/2} \tan x dx = \lim_{\epsilon \rightarrow 0+0} \int_0^{\pi/2-\epsilon} \frac{\sin x}{\cos x} dx = \lim_{\epsilon \rightarrow 0+0} \int_0^{\pi/2-\epsilon} (-1) \frac{(-\sin x)}{\cos x} dx =$

(the integrand has the form  $\frac{f'}{f}$ )

$$= \lim_{\epsilon \rightarrow 0+0} [-\ln(\cos x)]_0^{\pi/2-\epsilon} = \lim_{\epsilon \rightarrow 0+0} \left( -\ln \left( \cos \left( \frac{\pi}{2} - \epsilon \right) \right) + \ln(\cos 0) \right) = -(-\infty) + 0 = \infty$$

If  $\epsilon \rightarrow 0$ , then  $\cos \left( \frac{\pi}{2} - \epsilon \right) \rightarrow \cos \frac{\pi}{2} = 0$ ,  $\ln \left( \cos \left( \frac{\pi}{2} - \epsilon \right) \right) \rightarrow -\infty$



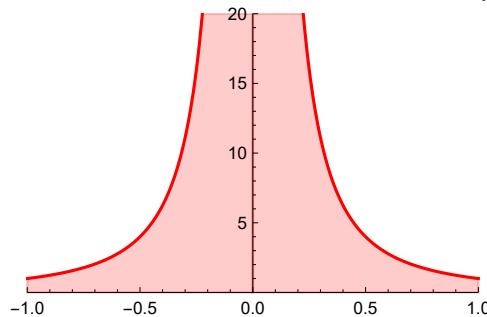
**Exercise 14.**  $\int_{-1}^1 x^{-2} dx = \int_{-1}^1 \frac{1}{x^2} dx = ?$

**Solution:** The integrand is not bounded:  $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$ .

The integral is divided into two parts:  $I = \int_{-1}^1 x^{-2} dx = \int_{-1}^0 x^{-2} dx + \int_0^1 x^{-2} dx = I_1 + I_2$

Since the integrand is even and the interval is symmetric about the origin then  $I_1 = I_2$ .

In Exercise 11. we have seen that  $I_2 = \int_0^1 \frac{1}{x^2} dx = \infty \implies I = \infty + \infty = \infty$ .



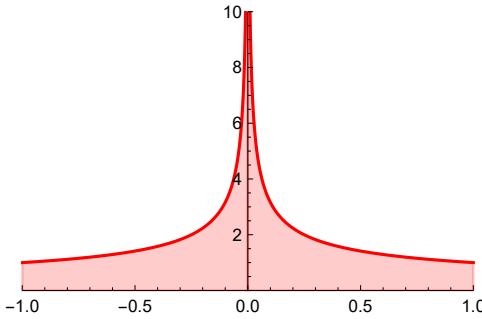
**Exercise 15.**  $\int_{-1}^1 x^{-1/2} dx = \int_{-1}^1 \frac{1}{\sqrt{x}} dx = ?$

**Solution:** The integrand is not bounded:  $\lim_{x \rightarrow 0} x^{-1/2} = \infty$ .

The integral is divided into two parts:  $I = \int_{-1}^1 x^{-1/2} dx = \int_{-1}^0 x^{-1/2} dx + \int_0^1 x^{-1/2} dx = I_1 + I_2$

Since the integrand is even and the interval is symmetric about the origin then  $I_1 = I_2$ .

In Exercise 10. we have seen that  $I_2 = \int_0^1 \frac{1}{\sqrt{x}} dx = 2 \implies I = I_1 + I_2 = 2 + 2 = 4$ .



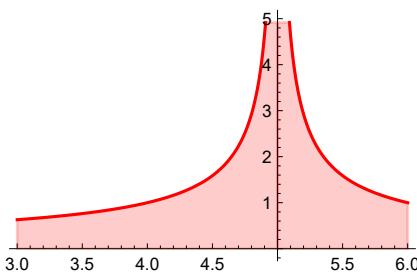
**Exercise 16.**  $\int_3^6 \frac{1}{\sqrt[3]{(x-5)^2}} dx = ?$

**Solution:** The integrand  $f(x) = \frac{1}{\sqrt[3]{(x-5)^2}}$  is not bounded:  $\lim_{x \rightarrow 5} \frac{1}{\sqrt[3]{(x-5)^2}} = \frac{1}{0^+} = \infty$

⇒ the integral is divided into two parts:

$$\begin{aligned} I &= \int_3^6 \frac{1}{\sqrt[3]{(x-5)^2}} dx = \int_3^5 \frac{1}{\sqrt[3]{(x-5)^2}} dx + \int_5^6 \frac{1}{\sqrt[3]{(x-5)^2}} dx = I_1 + I_2 \\ I_1 &= \lim_{\varepsilon_1 \rightarrow 0+0} \int_3^{5-\varepsilon_1} \frac{1}{\sqrt[3]{(x-5)^2}} dx = \lim_{\varepsilon_1 \rightarrow 0+0} \int_3^{5-\varepsilon_1} (x-5)^{-\frac{2}{3}} dx = \lim_{\varepsilon_1 \rightarrow 0+0} \left[ \frac{(x-5)^{\frac{1}{3}}}{\frac{1}{3}} \right]_3^{5-\varepsilon_1} = \lim_{\varepsilon_1 \rightarrow 0+0} \left[ 3 \sqrt[3]{x-5} \right]_3^{5-\varepsilon_1} = \\ &= \lim_{\varepsilon_1 \rightarrow 0+0} \left( 3 \sqrt[3]{(5-\varepsilon_1)-5} - 3 \sqrt[3]{3-5} \right) = \lim_{\varepsilon_1 \rightarrow 0+0} \left( 3 \sqrt[3]{-\varepsilon_1} - 3 \sqrt[3]{3-5} \right) = 0 - 3 \sqrt[3]{-2} = 3 \sqrt[3]{2} \\ I_2 &= \lim_{\varepsilon_2 \rightarrow 0+0} \int_{5+\varepsilon_2}^6 \frac{1}{\sqrt[3]{(x-5)^2}} dx = \dots = \lim_{\varepsilon_2 \rightarrow 0+0} \left[ 3 \sqrt[3]{x-5} \right]_{5+\varepsilon_2}^6 = \lim_{\varepsilon_2 \rightarrow 0+0} \left( 3 \sqrt[3]{6-5} - 3 \sqrt[3]{5+\varepsilon_2-5} \right) = \\ &= \lim_{\varepsilon_2 \rightarrow 0+0} \left( 3 \sqrt[3]{1} - 3 \sqrt[3]{\varepsilon_2} \right) = 3 - 0 = 3 \end{aligned}$$

$$I = I_1 + I_2 = 3 \sqrt[3]{2} + 3$$



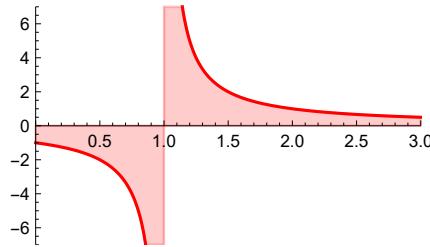
**Exercise 17.**  $\int_0^3 \frac{1}{x-1} dx = ?$

**Solution:** The integrand is not bounded at  $x = -1$ :

$$\lim_{x \rightarrow 1-0} \frac{1}{x-1} = -\infty, \quad \lim_{x \rightarrow 1+0} \frac{1}{x-1} = +\infty$$

$$I = \int_0^3 \frac{1}{x-1} dx = \int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx = I_1 + I_2$$

homework:  $I_1 = -\infty$ ,  $I_2 = +\infty \Rightarrow$  the sum doesn't exist, so the improper integral is divergent



**Exercise 18.**  $\int_{-\infty}^{\infty} e^{-|x|} dx = ?$

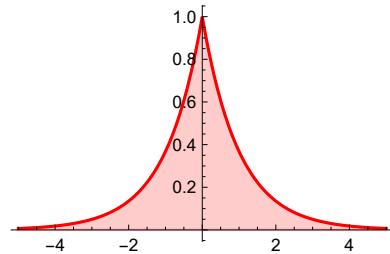
**Solution:**

$$I = \int_{-\infty}^{\infty} e^{-|x|} dx = \int_{-\infty}^0 e^{-|x|} dx + \int_0^{\infty} e^{-|x|} dx = \int_{-\infty}^0 e^{-(-x)} dx + \int_0^{\infty} e^{-x} dx = I_1 + I_2$$

$$\begin{aligned} \text{From exercise 3: } I_2 &= \int_0^{\infty} e^{-x} dx = \lim_{A \rightarrow \infty} \int_0^A e^{-x} dx = \lim_{A \rightarrow \infty} \left[ \frac{e^{-x}}{-1} \right]_0^A = \lim_{A \rightarrow \infty} [-e^{-x}]_0^A \\ &= \lim_{A \rightarrow \infty} (-e^{-A} - (-e^0)) = 0 + 1 = 1 \end{aligned}$$

The integrand,  $f(x) = e^{-|x|}$  is even and the interval is symmetric about the origin, so  $I_1 = I_2$

The solution is:  $I = I_1 + I_2 = 1 + 1 = 2$

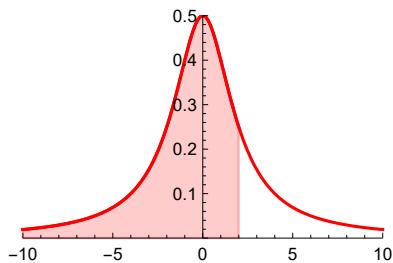


**Exercise 19.**  $\int_{-\infty}^2 \frac{2}{x^2 + 4} dx = ?$

**Solution:** First we calculate the indefinite integral:

$$\int \frac{2}{x^2 + 4} dx = \frac{2}{4} \int \frac{1}{(\frac{x}{2})^2 + 1} dx = \frac{1}{2} \frac{\arctg(\frac{x}{2})}{\frac{1}{2}} + c = \arctg\left(\frac{x}{2}\right) + c$$

$$\begin{aligned} \text{The improper integral is: } \int_{-\infty}^2 \frac{2}{x^2 + 4} dx &= \lim_{B \rightarrow \infty} \int_B^2 \frac{2}{x^2 + 4} dx = \lim_{B \rightarrow \infty} \left[ \arctg\left(\frac{x}{2}\right) \right]_B^2 = \\ &= \lim_{B \rightarrow \infty} \left( \arctg\left(\frac{2}{2}\right) - \arctg\left(\frac{B}{2}\right) \right) = \arctg 1 - \left( -\frac{\pi}{2} \right) = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4} \end{aligned}$$



**Exercise 20.**  $\int_{-\infty}^0 \frac{1}{(3x-2)^2} dx = ?$

**Solution:**  $I = \int_{-\infty}^0 \frac{1}{(3x-2)^2} dx = \lim_{A \rightarrow -\infty} \int_A^0 \frac{1}{(3x-2)^2} dx = \lim_{A \rightarrow -\infty} \int_A^0 (3x-2)^{-2} dx =$   
 $= \lim_{A \rightarrow -\infty} \left[ \frac{(3x-2)^{-1}}{-1 \cdot 3} \right]_A^0 = \lim_{A \rightarrow -\infty} \left[ -\frac{1}{3} \frac{1}{3x-2} \right]_A^0 = \lim_{A \rightarrow -\infty} \left( -\frac{1}{3} \frac{1}{0-2} + \frac{1}{3} \frac{1}{3A-2} \right) = \frac{1}{6} + 0 = \frac{1}{6}$

