24th and 25th lectures

Definite integral

The Riemann integral

Definition. A partition of an interval $[a, b]$ is a finite set $P = \{x_0, x_1, \ldots x_n\}$ such that

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

Definition. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and $P = \{x_0, x_1, \ldots x_n\}$ is a partition of $[a, b]$. Let

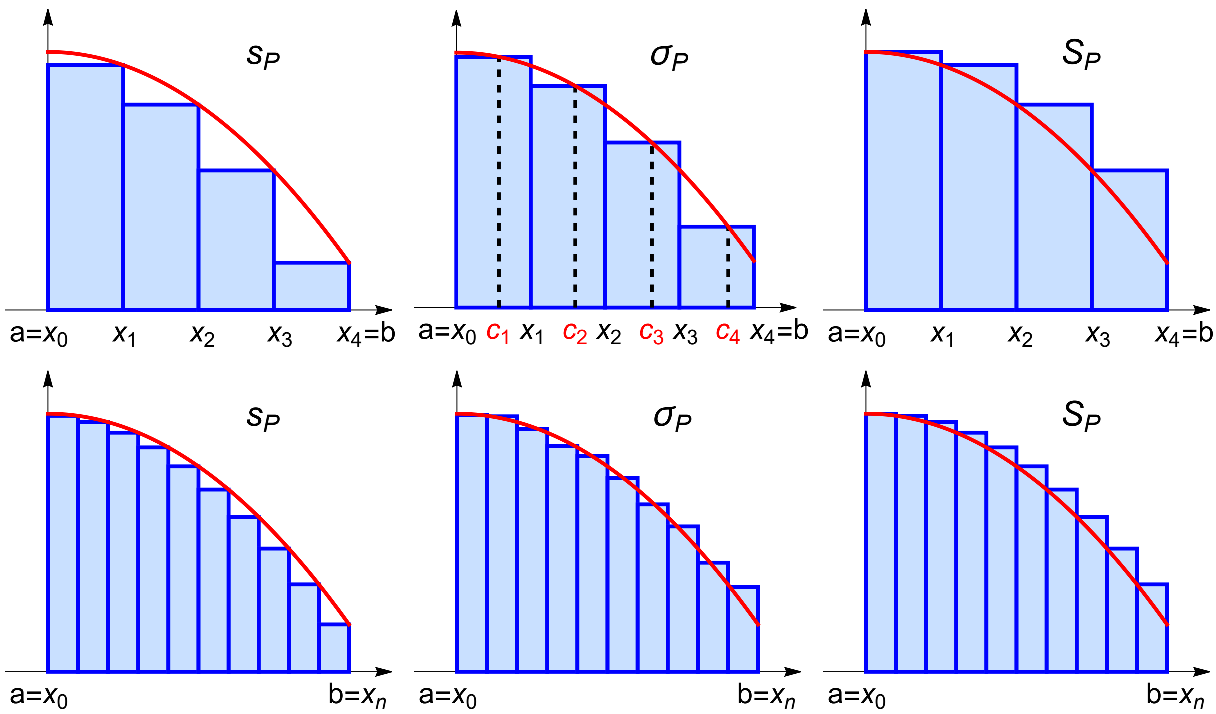
$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

The **lower Darboux sum** of f with respect to P is $s_P = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$.

The **upper Darboux sum** of f with respect to P is $S_P = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$.

The **Riemann sum** of f with respect to P is $\sigma_P = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$, where

$c_k \in [x_{k-1}, x_k]$ is arbitrary. The points c_k are called the **evaluation points**.



Statement. $s_P \le \sigma_P \le S_P$ for all partitions P.

Proof. It follows from the fact that $m_k \le f(c_k) \le M_k$ on each subinterval $[x_{k-1}, x_k]$.

Definition. Let P_1 and P_2 be partitions of $[a, b]$. If P_2 contains all points of P_1 and some additional points then P_2 is a refinement of P_1.

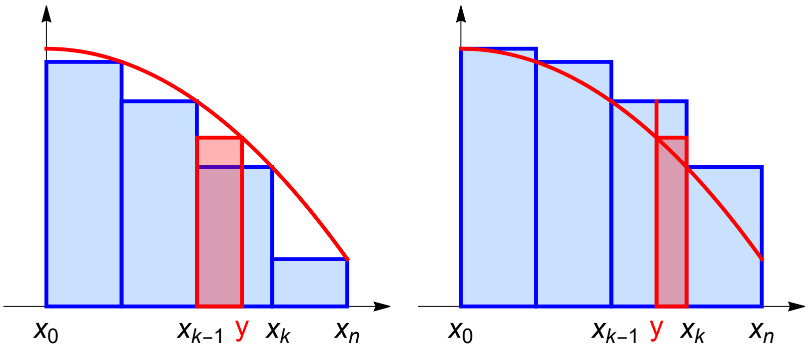
Theorem. If P_2 is a refinement of P_1 then $s_{P_1} \le s_{P_2}$ and $S_{P_1} \le S_{P_2}$,
that is, by refining a partition, the lower Darboux sum cannot decrease and the upper Darboux sum cannot increase.

Proof. Let P_2 be the partition that is obtained from $P_1 = \{x_0, x_1, \ldots, x_n\}$ by adding the point $x_{k-1} < y < x_k$. We prove $s_{P_1} \le s_{P_2}$.

Let $A = \inf\{f(x) : x \in [x_{k-1}, y]\}$ and $B = \inf\{f(x) : x \in [y, x_k]\}$.

Then $m_k(x_k - x_{k-1}) = m_k(y - x_{k-1}) + m_k(x_k - y) \le A(y - x_{k-1}) + B(x_k - y)$

$\Longrightarrow s_{P_2} - s_{P_1} = A(y - x_{k-1}) + B(x_k - y) - m_k(x_k - x_{k-1}) \ge 0.$



Theorem. $s_{P_1} \le S_{P_2}$ for any partitions P_1 and P_2 of $[a, b]$, that is,
any lower Darboux sum is less than or equal to any upper Darboux sum.

Proof. Let $P_3 = P_1 \cup P_2 \Longrightarrow P_3$ is a refinement of P_1 and $P_2 \Longrightarrow s_{P_1} \le s_{P_3} \le S_{P_3} \le S_{P_2}$

Definition. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded.

The **lower Darboux integral** of f is $\underline{\int_a^b} f = \sup\{s_P : P \text{ is a partition of } [a, b]\}$.

The **upper Darboux integral** of f is $\overline{\int_a^b} f = \inf\{S_P : P \text{ is a partition of } [a, b]\}$.

Consequence: $\underline{\int_a^b} f \le \overline{\int_a^b} f$

Definition. If $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and $I = \underline{\int_a^b} f = \overline{\int_a^b} f$ then f is **Riemann integrable** on $[a, b]$.

In this case the Riemann integral of f on $[a, b]$ is denoted as

$I = \int_a^b f(x)\,dx$ or $I = \int_a^b f.$ (f is called the integrand.)

Notation. $R[a, b]$ denotes the set of those functions that are Riemann integrable on $[a, b]$

Remark. If $f : [a, b] \longrightarrow \mathbb{R}$ is not bounded on $[a, b]$ or bounded but $\underline{\int_a^b} f < \overline{\int_a^b} f$ then f is not Riemann integrable on $[a, b]$.

Example: Let $f(x) = c \in \mathbb{R}$, $\int_a^b c\,dx = ?$

$$s_P = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} c(x_k - x_{k-1}) = c(b - a),$$

$$S_P = \sum_{k=1}^{n} M_k (x_k - x_{k-1}) = \sum_{k=1}^{n} c (x_k - x_{k-1}) = c(b-a) \text{ for all partitions } P.$$

$$\underline{\int_a^b} f = \sup\{s_P\} = c(b-a) = \inf\{S_P\} = \overline{\int_a^b} f \implies \int_a^b c\, dx = c(b-a)$$

Example: The Dirichlet function $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$ is bounded, and for all partitions P of $[0, 1]$, $s_P = 0$ and $S_P = 1$

$$\implies \underline{\int_a^b} f = 0 \text{ and } \overline{\int_a^b} f = 1$$

$\implies f$ is not integrable on $[0, 1]$.

Necessary and sufficient conditions for Riemann integrability

Definition. The **mesh** or **norm of a partition** is the maximal distance between adjacent points in the partition: $\Delta P = \max_{k \in \{1,\ldots,n\}} (x_k - x_{k-1})$.

Statement. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and (P_n) is a sequence of partitions of $[a, b]$.

If $\lim_{n\to\infty} \Delta P_n = 0$ then $\lim_{n\to\infty} s_{P_n} = \underline{\int_a^b} f$ and $\lim_{n\to\infty} S_{P_n} = \overline{\int_a^b} f$

Statement. a) If $\exists \int_a^b f(x)\, dx \implies$ for all partition sequences (P_n) for which $\lim_{n\to\infty} \Delta P_n = 0$:

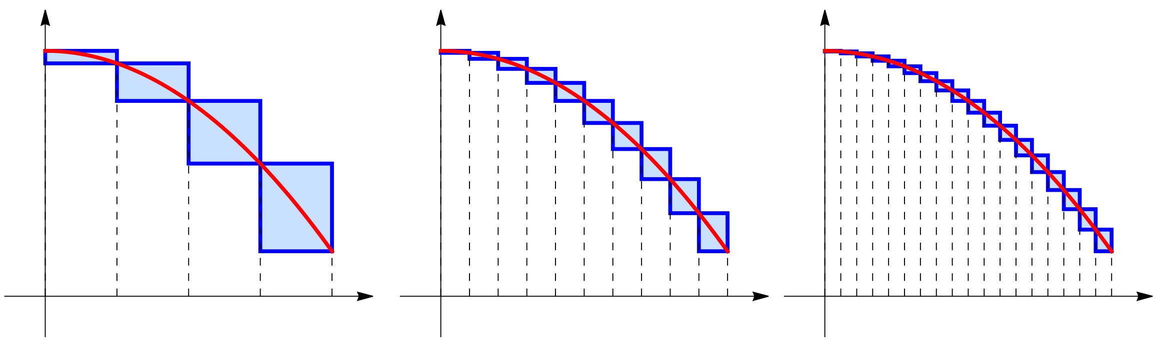
$$\lim_{n\to\infty} s_{P_n} = \lim_{n\to\infty} S_{P_n} = \int_a^b f(x)\, dx.$$

b) If (P_n) is a partition sequence for which $\lim_{n\to\infty} \Delta P_n = 0$ and $\lim_{n\to\infty} s_{P_n} = \lim_{n\to\infty} S_{P_n} = I$

$$\implies \exists \int_a^b f(x)\, dx = I.$$

Definition. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and $P = \{x_0, x_1, \ldots x_n\}$ is a partition of $[a, b]$. Then the oscillation sum of f related to the partition P is

$$O_P = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = S_P - s_P.$$



Theorem (Riemann's criterion for integrability). Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded.
f is integrable on $[a, b]$ $\iff$ for all $\varepsilon > 0$ there exists a partition P such that $O_P = S_P - s_P < \varepsilon$.

Proof. $\implies$: Assume that f is integrable and $\varepsilon > 0$. Then there exist partitions P_1 and P_2 such that

$$0 \le S_{P_2} - \overline{\int_a^b} f < \frac{\varepsilon}{2} \text{ and } 0 \le \underline{\int_a^b} f - s_{P_1} < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$ (P is a common refinement of P_1 and P_2). Then $s_{P_1} \le s_P \le S_P \le S_{P_2}$, so

$$0 \le O_P = S_P - s_P \le S_{P_2} - s_{P_1} = \left(S_{P_2} - \overline{\int_a^b}\right) + \left(\underline{\int_a^b} f - s_{P_1}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Longleftarrow$: For any partition P, $s_P \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le S_P$, so

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le S_P - s_P = O_P < \varepsilon \text{ for all } \varepsilon > 0 \Longrightarrow \overline{\int_a^b} f = \underline{\int_a^b} f,$$ that is, f is integrable.

Remark. Recall that the **Riemann sum** of f with respect to the partition P is

$\sigma_P = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$, where the evaluation points $c_k \in [x_{k-1}, x_k]$ are arbitrary and

$s_P \le \sigma_P \le S_P$ for all partitions P.

Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is bounded. Then

1. $\exists \int_a^b f(x)\,dx = I \Longrightarrow$ for all partition sequences (P_n) for which $\lim_{n\to\infty} \Delta P_n = 0$:

$\lim_{n\to\infty} \sigma_{P_n} = \int_a^b f(x)\,dx = I$ (independent of the choice of the evaluation points).

2. $\exists \int_a^b f(x)\,dx = I \Longleftarrow$ there exists a partition sequence (P_n) for which $\lim_{n\to\infty} \Delta P_n = 0$

and $\exists \lim_{n\to\infty} \sigma_{P_n} = I$ (independent of the choice of the evaluation points).

Remark. The proof of part 1. is obvious, since $s_{P_n} \le \sigma_{P_n} \le S_{P_n}$ and $\lim_{n\to\infty} s_{P_n} = \lim_{n\to\infty} S_{P_n} = I$.

Remark. It is important that the limit exists independent of the choice of $c_k \in [x_{k-1}, x_k]$ in the Riemann sum. For example, assume that f is the Dirichlet function on $[a, b]$ and (P_n) is a sequence of partitions for which $\lim_{n\to\infty} \Delta P_n = 0$.

If c_k is rational: $\sigma_{P_n} = \sum_{k=1}^{n} 1 \cdot (x_k - x_{k-1}) = 1 \cdot (b - a) \longrightarrow b - a$

If c_k is irrational: $\sigma_{P_n} = \sum_{k=1}^{n} 0 \cdot (x_k - x_{k-1}) = 0 \longrightarrow 0$

$\Longrightarrow$ the Dirichlet function is not integrable on any interval.

Sufficient conditions for Riemann integrability

Theorem. If f is monotonic and bounded on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Proof. Assume that f is **monotonically increasing**.

1) If $f(a) = f(b)$ then f is constant, so $f \in R[a, b]$.
2) If $f(a) < f(b)$ then we show that for all $\varepsilon > 0$ there exists a partition P such that the oscillation sum $O_P = S_P - s_P < \varepsilon$.
3) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition with mesh

$$\Delta P = \max_{k \in \{1,\ldots,n\}} (x_k - x_{k-1}) < \delta = \frac{\varepsilon}{f(b) - f(a)} > 0.$$

4) Then for the oscillation sum we get that

$$O_P = S_P - s_P = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}) <$$

$$< \delta \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = \delta(f(b) - f(a)) = \varepsilon.$$

Theorem. If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous then f is Riemann integrable on $[a, b]$.

Proof. 1) We prove that for all $\varepsilon > 0$ there exists a partition P such that the oscillation sum $O_P = S_P - s_P < \varepsilon$.

2) f is continuous on $[a, b]$ $\Longrightarrow$ f is bounded and also uniformly continuous on $[a, b]$.

$\Longrightarrow$ for $\dfrac{\varepsilon}{b-a} > 0$ there exists $\delta > 0$ such that $\forall\, x, y \in [a, b]$,

$$|x - y| < \delta \Longrightarrow \left| f(x) - f(y) \right| < \frac{\varepsilon}{b-a}.$$

3) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition with mesh $\Delta P = \max\limits_{k \in \{1,\ldots,n\}} (x_k - x_{k-1}) < \delta$.

4) f is continuous on $[x_{k-1}, x_k]$ $\Longrightarrow$ by the extreme value theorem f has a minimum for some $c_k \in [x_{k-1}, x_k]$ and a maximum for some $d_k \in [x_{k-1}, x_k]$, let $f(c_k) = m_k$, $f(d_k) = M_k$.

5) Then obviously $|d_k - c_k| < \delta$, so for the oscillation sum we get that

$$O_P = S_P - s_P = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} (f(d_k) - f(c_k))(x_k - x_{k-1}) =$$

$$= \sum_{k=1}^{n} \left| f(d_k) - f(c_k) \right| (x_k - x_{k-1}) < \sum_{k=1}^{n} \frac{\varepsilon}{b-a} (x_k - x_{k-1}) =$$

$$= \frac{\varepsilon}{b-a} \sum_{k=1}^{n} (x_k - x_{k-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Theorem. If $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and continuous except finitely many points then f is Riemann integrable on $[a, b]$.

Proof. 1) We prove it in the case of one point. Let $c \in [a, b]$ and assume that f is continuous on $[a, b] \setminus \{c\}$. Let $K > 0$ be such that $|f(x)| \le K$ for all $x \in [a, b]$. We show that for all $\varepsilon > 0$ there exists a partition P such that $O_P < \varepsilon$.

2) If $c - \dfrac{\varepsilon}{8K} > a$ then let $c_1 = c - \dfrac{\varepsilon}{8K}$ and let P_1 be a partition of $[a, c_1]$ such that $O_{P_1} < \dfrac{\varepsilon}{4}$.

Such a partition exists since f is continuous on $[a, c_1]$.

If $c - \dfrac{\varepsilon}{8K} \le a$ then let $c_1 = a$ and $P_1 = \{a\}$.

3) If $c + \dfrac{\varepsilon}{8K} < b$ then let $c_2 = c + \dfrac{\varepsilon}{8K}$ and let P_2 be a partition of $[c_2, b]$ such that $O_{P_2} < \dfrac{\varepsilon}{4}$.

Such a partition exists since f is continuous on $[c_2, b]$.

If $c + \dfrac{\varepsilon}{8K} \ge b$ then let $c_2 = b$ and $P_2 = \{b\}$.

4) Then $P = P_1 \cup P_2$ is a suitable choice.

Remark. If $f, g : [a, b] \longrightarrow \mathbb{R}$, f is Riemann integrable and $f(x) = g(x)$ except finitely many points in $[a, b]$ then g is Riemann integrable and $\int_a^b f = \int_a^b g$.

Newton-Leibniz formula

Theorem (First fundamental theorem of calculus, Newton-Leibniz formula).

If $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable and $F : [a, b] \longrightarrow \mathbb{R}$ is an antiderivative of f, that is, $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,\mathrm{dx} = F(b) - F(a) = [F(x)]_a^b$$

Proof. Let (P_n) be a partition sequence of $[a, b]$ such that $\lim\limits_{n\to\infty} \Delta P_n = 0$.

For all $k \in \{1, 2, \ldots, n\}$, F is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k), so by Lagrange's mean value theorem there exists $x_{k-1} < c_k < x_k$ such that

$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(c_k) = f(c_k) \Longrightarrow F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1})$$

$$\Longrightarrow F(b) - F(a) = (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \ldots + (F(x_n) - F(x_{n-1})) =$$

$$= \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = \sigma_{P_n}$$

$$\Longrightarrow F(b) - F(a) = \sigma_{P_n}$$

Taking the limits of both sides: $\lim\limits_{n\to\infty} (F(b) - F(a)) = \lim\limits_{n\to\infty} \sigma_{P_n}$

The left-hand side is independent of n and since f is integrable then the limit of the right-hand side is the integral of f, so

$F(b) - F(a) = \int_a^b f(x)\,\mathrm{dx}$.

Remark. The geometrical meaning of $\int_a^b f$ is the signed area under the graph of f on $[a, b]$.

Remark. Both conditions of the theorem are important as the following examples show.

Examples

Example 1. Let $F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, then $F'(x) = f(x) = \begin{cases} 2\,x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

f has an antiderivative, however, $\int_0^1 f(x)\,\mathrm{dx}$ doesn't exist, since f is not bounded.

Example 1. $\int_0^5 \operatorname{sign}(x^2 - 5\,x + 6)\,\mathrm{dx}$ exists, since f is continuous except 2 points. However, by Darboux's theorem, f doesn't have an antiderivative, since f has jump discontinuities.

Properties of Riemann integrable functions

Definition. If $f \in R[a, b]$ $\int_b^a f(x)\,\mathrm{dx} := -\int_a^b f(x)\,\mathrm{dx}$, $\int_a^a f(x)\,\mathrm{dx} := 0$

Theorem. Let $f, g \in R[a, b]$ and $\lambda \in \mathbb{R}$. Then

(1) $\lambda f,\ f + g,\ f - g \in R[a, b]$ and $\int_a^b \lambda f = \lambda \int_a^b f$, $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$

(2) $[\alpha, \beta] \subset [a, b] \Longrightarrow f \in R[\alpha, \beta]$

(3) $a < c < b \implies \int_a^b f = \int_a^c f + \int_c^b f$

(4) $f(x) \leq g(x)\ \forall\, x \in [a, b] \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

(5) $|f| \in R[a, b] \implies \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

(6) $\inf_{[a,b]} f \leq \frac{1}{b-a} \int_a^b f \leq \sup_{[a,b]} f$

Integration by parts

Theorem. If f and g are continuously differentiable on $[a, b]$ then $\int_a^b f' g = [f\,g]_a^b - \int_a^b f\,g'$

Integration by substitution

Theorem. If g is continuously differentiable, strictly monotonic, $[a, b] \subset D_g$ and

f is continuous on $[a, b]$ then $\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\,g'(t)\,dt.$

Example. $I = \int_0^{\ln 2} \sqrt{e^x - 1}\,dx = ?$

Solution. Substitution: $t = \sqrt{e^x - 1} \implies x = x(t) = \ln\left(t^2 + 1\right)$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t^2+1} \cdot 2\,t \implies dx = \frac{2\,t}{t^2+1}\,dt$$

The bounds will change: $x_1 = 0 \implies t_1 = \sqrt{e^0 - 1} = 0$

$$x_2 = \ln 2 \implies t_2 = \sqrt{e^{\ln 2} - 1} = \sqrt{2-1} = 1$$

$$I = \int_0^{\ln 2} \sqrt{e^x - 1}\,dx = \int_{t_1}^{t_2} t \cdot \frac{2\,t}{t^2+1}\,dt = \int_0^1 \frac{2\,t^2}{t^2+1}\,dt = \int_0^1 \frac{2\left(t^2+1\right) - 2}{t^2+1}\,dt = \int_0^1 \left(2 - \frac{2}{t^2+1}\right) dt =$$

$$= [2\,t - 2\,\mathrm{arctg}\,t]_0^1 = (2 \cdot 1 - 2\,\mathrm{arctg}1) - (0 - 0) = 2 - \frac{\pi}{2}$$

Lebesgue's theorem

Definition. We say that the set $A \subset \mathbb{R}$ has **Lebesgue measure 0** if for all $\varepsilon > 0$ there exist

sequences (x_n) and (y_n) such that $x_n \leq y_n$, $A \subset \bigcup_{n=1}^{\infty} [x_n, y_n]$ and $\sum_{n=1}^{\infty} (y_n - x_n) < \varepsilon$.

(That is, A can be covered with countably many intervals such that their total length is less than ε.)

Examples. 1) Any countable set of $\mathbb{R}$ has Lebesgue measure 0, for example $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$.

2) The Cantor set is defined in the following way. Let $C_0 = [0, 1]$.

C_1 is obtained from C_0 by deleting the open middle third from C_0, that is,

$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$.

C_2 is obtained from C_1 by deleting the open middle thirds from C_1, that is,

$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$

Continuing this process, C_{n+1} is obtained from C_n by deleting the open middle thirds of each of these intervals. The Cantor set is $C = \bigcap_{n \in \mathbb{N}} C_n$.

It can proved that the Cantor set is uncountable but has Lebesgue measure 0.

Theorem (Lebesgue). The function $f : [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if it is bounded and the set of discontinuities of f has Lebesgue measure 0.

Remark. If $f : [a, b] \longrightarrow \mathbb{R}$ is monotonic then f has at most countably many discontinuities (and they are jump discontinuities), so by Lebesgue's theorem f is Riemann integrable.

Example*. The Riemann function is defined as

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p \in \mathbb{Z},\ \text{and } q \in \mathbb{N}^+ \text{ are coprimes} \end{cases}$$

Prove that

a) $\lim_{x \to a} f(x) = 0\ \ \forall\, a \in \mathbb{R}$;

a) f is continuous at all irrational numbers;

b) f is discontinuous at all rational numbers.

Solution. If $q \in \mathbb{N}^+$ is fixed then the set $\mathbb{Z} \cdot \frac{1}{q} = \left\{\frac{k}{q} : k \in \mathbb{Z}\right\}$ does not have any real limit points.

Therefore a finite union of such sets, $A_n = \left\{\frac{p}{q} : p \in \mathbb{Z},\ q \in \{1, 2, ..., n\}\right\}$ does not have any limit points either. If $x \in \mathbb{R} \setminus A_n$ the $|f(x)| < \frac{1}{n}$, so for all $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = 0$.

$\Longrightarrow$ f is continuous at all irrational points and has a removable discontinuity at all rational points.

The Riemann function is bounded and the set of discontinuities is countable, so it has Lebesgue measure 0 $\Longrightarrow$ f is Riemann integrable and $\int_a^b f(x)\, dx = 0$.

The integral function

Definition. Assume that f is Riemann integrable on $[a, b]$. Then the function

$$F(x) = \int_a^x f(t)\, dt,\ \ x \in [a, b]$$

is called the **integral function** of f.

Theorem (Second fundamental theorem of calculus).

Assume that f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$. Then

1. F is Lipschitz continuous on $[a, b]$.
2. If f is continuous at $x_0 \in [a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof. 1. Let $K = \sup_{[a,b]} |f(x)|$. If $K = 0$ then $f = 0$ so $F = 0$ is Lipschitz continuous.

If $K \neq 0$ then $0 < K \in \mathbb{R}$. Let $\varepsilon > 0$ and $\delta(\varepsilon) = \frac{\varepsilon}{K}$. If $x, y \in [a, b]$ such that $|x - y| < \delta$ then

$$|F(x) - F(y)| = \left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right| = \left| \int_y^x f(t)\,dt \right| \leq \left| \int_y^x |f(t)|\,dt \right| \leq \left| \int_y^x K\,dt \right| \leq$$

$$\leq K\,|x - y| < K\,\delta = \varepsilon \implies F \text{ is Lipschitz continuous.}$$

2. $F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon \text{ if } 0 < |x - x_0| < \delta.$$

Let $\varepsilon > 0$. Since f is continuous at x_0 then $\exists\, \delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$. Then with this δ

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{F(x) - F(x_0) - f(x_0)(x - x_0)}{x - x_0} \right| = \left| \frac{\int_{x_0}^x f(t)\,dt - \int_{x_0}^x f(x_0)\,dt}{x - x_0} \right| =$$

$$= \left| \frac{\int_{x_0}^x (f(t) - f(x_0))\,dt}{x - x_0} \right| \leq \frac{\left| \int_{x_0}^x |f(t) - f(x_0)|\,dt \right|}{|x - x_0|} \leq \frac{\left| \int_{x_0}^x \varepsilon\,dt \right|}{|x - x_0|} = \frac{|\varepsilon(x - x_0)|}{|x - x_0|} = \varepsilon.$$

Consequence.

1. If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$ then $F'(x) = f(x)$ $\forall\, x \in [a, b]$.
2. Every continuous function has an antiderivative.

Examples

Example 1. Calculate the derivatives of the following functions:

a) $F(x) = \int_0^x \sin t^2\,dt$, $x \neq 0$ b) $G(x) = \int_0^{x^3} \sin t^2\,dt$ c) $H(x) = \int_{x^2}^{x^3} \sin t^2\,dt$

Solution. a) $F'(x) = \sin x^2$, since $f(t) = \sin(t^2)$ is continuous.

b) $G(x) = F(x^3) \implies G'(x) = F'(x^3) \cdot 3x^2 = \sin((x^3)^2) \cdot 3x^2 = \sin(x^6) \cdot 3x^2$

c) $H(x) = \int_0^{x^3} \sin t^2\,dt - \int_0^{x^2} \sin t^2\,dt = F(x^3) - F(x^2) \implies H'(x) = \sin(x^6) \cdot 3x^2 - \sin(x^4) \cdot 2x$

Example 2. $\lim_{x \to 0} \frac{\int_0^x \arctan t^2\,dt}{x^2} = ?$

Solution. The limit has the form $\frac{0}{0}$ and the numerator is differentiable since

$f(t) = \arctan t^2$ is continuous

$$\Longrightarrow \lim_{x\to 0}\frac{\int_0^x \arctan t^2\,dt}{x^2} \overset{L'H}{=} \lim_{x\to 0}\frac{\arctan x^2}{2x} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{1+x^4}\cdot 2x}{2} = 0$$

Improper integrals

Case 1: The interval is not bounded

Definition. Let $a \in \mathbb{R}$ and assume that f is Riemann integrable on $[a, b]$ for all $b \geq a$.

If the limit $\lim_{b\to\infty}\int_a^b f(x)\,dx \in \mathbb{R}$ exists then we say that f is **improperly integrable** or f has an **improper integral** on $[a, \infty)$ and the value of the integral is

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

In this case we also say that **the improper integral converges**.

If the limit $\lim_{b\to\infty}\int_a^b f(x)\,dx$ doesn't exist or if $\lim_{b\to\infty}\int_a^b f(x)\,dx = \infty$ or $-\infty$ then we say that f is not improperly integrable on $[a, \infty)$ or **the improper integral diverges**.

Definition. Similarly, let $b \in \mathbb{R}$ and assume that f is Riemann integrable on $[a, b]$ for all $a \leq b$.

Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

If the limit exists and is finite then the improper integral converges.

If the limit doesn't exist or exists but is ∞ or $-\infty$ then the improper integral diverges.

Examples

Exercise 1. $I = \int_2^\infty \frac{6}{x^2+x-2}\,dx = ?$

Solution. Using partial fraction decomposition:

$$\frac{6}{x^2+x-2} = \frac{6}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \Longrightarrow 6 = A(x+2) + B(x-1)$$

If $x = -2$: $B = -2$

If $x = 1$: $A = 2$

$$I = \lim_{b\to\infty}\int_2^b \frac{6}{x^2+x-2}\,dx = \lim_{b\to\infty}\int_2^b\left(\frac{2}{x-1} - \frac{2}{x+2}\right)dx =$$

$$= 2\lim_{b\to\infty}\left[\ln(x-1) - \ln(x+2)\right]_2^b = 2\lim_{b\to\infty}\left(\ln(b-1) - \ln(b+2) - (\ln 1 - \ln 4)\right) =$$

$$= 2\lim_{b\to\infty}\left(\ln\frac{b-1}{b+2} + \ln 4\right) = 2\cdot(0 + \ln 4) = 2\ln 4$$

(the improper integral converges)

Exercise 2. $I = \int_{-\infty}^{-1} \frac{1}{(x-2)\sqrt{\ln(2-x)}} \, dx = ?$

Solution. We use that $\int f' f^{\alpha} = \frac{f^{\alpha+1}}{\alpha+1} + c, \ \alpha \neq -1$:

$$I = \lim_{a \to -\infty} \int_a^{-1} \frac{-1}{2-x} (\ln(2-x))^{-\frac{1}{2}} \, dx =$$

$$= \lim_{a \to -\infty} \left[\frac{(\ln(2-x))^{\frac{1}{2}}}{\frac{1}{2}} \right]_a^{-1} = 2 \lim_{a \to -\infty} \left(\sqrt{\ln 3} - \sqrt{\ln(2-a)} \right) = -\infty$$

(the improper integral diverges)

Important remark

Definition. Let $a, b \in \overline{\mathbb{R}}$. The improper integral $I = \int_a^b f(x) \, dx$ is said to be convergent if for all $c \in (a, b)$ the improper integrals

$$I_1 = \int_a^c f(x) \, dx \ \text{ and } \ I_2 = \int_c^b f(x) \, dx$$

are both convergent.
The improper integral I is divergent if at least one of I_1 and I_2 is divergent.

Definition. $\int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f(x) \, dx$ if the double limit exists and is finite.

Remark. Because of the previous definition $\int_{-\infty}^{\infty} f(x) \, dx \neq \lim_{a \to \infty} \int_{-a}^{a} f(x) \, dx$.

For example, $\int_{-\infty}^{\infty} x \, dx$ is divergent, since $\int_{-\infty}^{0} x \, dx = -\infty$ and $\int_0^{\infty} x \, dx = \infty$.

However, $\lim_{a \to \infty} \int_{-a}^{a} x \, dx = \lim_{a \to \infty} \left[\frac{x^2}{2} \right]_{-a}^{a} = \lim_{a \to \infty} \left(\frac{a^2}{2} - \frac{a^2}{2} \right) = 0.$

Case 2: The function is not bounded

Definition. Assume that f is not bounded at a but f is Riemann integrable on $[A, b]$ for all $a < A \leq b$.
Then

$$\int_a^b f(x) \, dx = \lim_{\delta \to +0} \int_{a+\delta}^{b} f(x) \, dx \quad \text{or} \quad \int_a^b f(x) \, dx = \lim_{A \to a+0} \int_A^b f(x) \, dx$$

Definition. Assume that f is not bounded at b but f is Riemann integrable on $[a, B]$ for all $a \leq B < b$.
Then

$$\int_a^b f(x) \, dx = \lim_{\delta \to +0} \int_a^{b-\delta} f(x) \, dx \quad \text{or} \quad \int_a^b f(x) \, dx = \lim_{B \to b-0} \int_a^B f(x) \, dx$$

Definition. If f is not bounded at $c \in (a, b)$ then
$$\int_a^b f(x)\,\mathrm{dx} = \int_a^c f(x)\,\mathrm{dx} + \int_c^b f(x)\,\mathrm{dx} = \lim_{\delta_1 \to +0} \int_a^{c-\delta_1} f(x)\,\mathrm{dx} + \lim_{\delta_2 \to +0} \int_{c+\delta_2}^b f(x)\,\mathrm{dx}$$

Examples

Exercise 1. $I = \int_5^7 \frac{1}{\sqrt[3]{(x-5)^2}}\,\mathrm{dx} = ?$

Solution. $I = \lim_{\delta \to +0} \int_{5+\delta}^7 (x-5)^{-\frac{2}{3}}\,\mathrm{dx} = \lim_{\delta \to +0} \left[\frac{(x-5)^{\frac{1}{3}}}{\frac{1}{3}}\right]_{5+\delta}^7 = 3 \lim_{\delta \to +0} \left(\sqrt[3]{2} - \sqrt[3]{\delta}\right) = 3\sqrt[3]{2}.$

Exercise 2. $I = \int_0^1 \frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}}\,\mathrm{dx} = ?$

Solution. We use that $\int f' f^\alpha = \frac{f^{\alpha+1}}{\alpha+1} + c, \ \alpha \neq -1$:

$$I = \lim_{\delta \to +0} \int_0^{1-\delta} \frac{1}{\sqrt{1-x^2}} (\arcsin x)^{\frac{1}{2}}\,\mathrm{dx} = \lim_{\delta \to +0} \left[\frac{(\arcsin x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{1-\delta} = \frac{2}{3} \lim_{\delta \to +0} \left((\arcsin(1-\delta))^{\frac{3}{2}} - 0\right) = \frac{2}{3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}}$$

Improper integrals of $f(x) = \frac{1}{x^\alpha}$

Statement. The improper integral $\int_1^\infty \frac{1}{x^\alpha}\,\mathrm{dx}$ is convergent if $\alpha > 1$ and divergent if $\alpha \leq 1$.

Proof.

If $\alpha = 1$, then $\int_1^\infty \frac{1}{x^\alpha}\,dx = \lim_{A \to \infty} \int_1^A \frac{1}{x}\,dx = \lim_{A \to \infty} [\ln x]_1^A = \lim_{A \to \infty} (\ln A - \ln 1) = \infty$

If $\alpha \neq 1$, then

$$\int_1^\infty \frac{1}{x^\alpha}\,dx = \lim_{A \to \infty} \int_1^A x^{-\alpha}\,dx = \lim_{A \to \infty} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^A = \lim_{A \to \infty} \left(\frac{A^{-\alpha+1}}{-\alpha+1} - \frac{1}{-\alpha+1}\right) = \begin{cases} 0 - \frac{1}{-\alpha+1} = \frac{1}{\alpha-1}, & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha < 1 \end{cases}$$

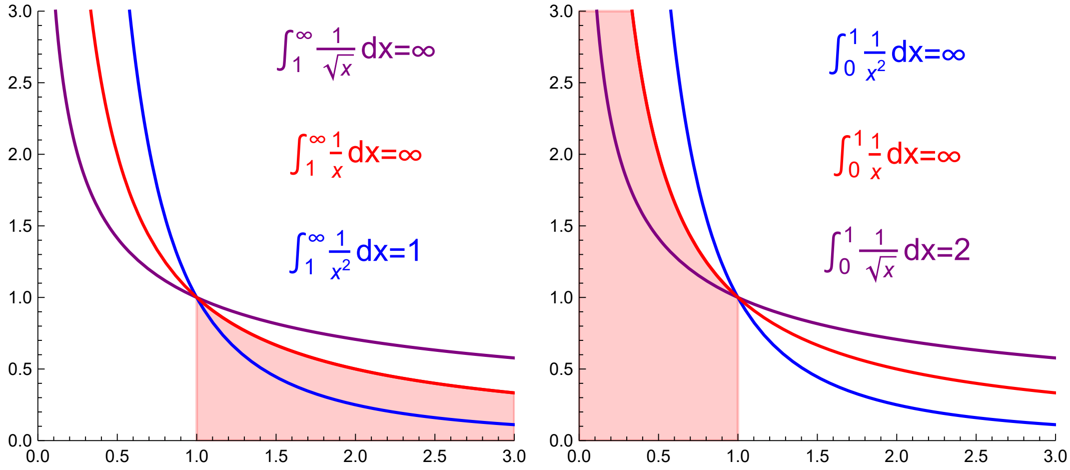
Statement. The improper integral $\int_0^1 \frac{1}{x^\alpha}\,\mathrm{dx}$ is convergent if $\alpha < 1$ and divergent if $\alpha \geq 1$.

Proof.

If $\alpha = 1$, then $\int_0^1 \frac{1}{x^\alpha}\,dx = \lim_{\varepsilon \to 0+0} \int_{0+\varepsilon}^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to 0+0} [\ln x]_\varepsilon^1 = \lim_{\varepsilon \to 0+0} (\ln 1 - \ln \varepsilon) = 0 - (-\infty) = \infty$

If $\alpha \neq 1$, then $\int_0^1 \frac{1}{x^\alpha}\,dx = \lim_{\varepsilon \to 0+0} \int_{0+\varepsilon}^1 x^{-\alpha}\,dx = \lim_{\varepsilon \to 0+0} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_\varepsilon^1 = \lim_{\varepsilon \to 0+0} \left(\frac{1}{-\alpha+1} - \frac{\varepsilon^{-\alpha+1}}{-\alpha+1}\right) =$

$$= \begin{cases} \infty, & \text{if } \alpha > 1 \\ \frac{1}{-\alpha+1} - 0 = \frac{1}{1-\alpha}, & \text{if } \alpha < 1 \end{cases}$$



Comparison test for improper integrals

Theorem. Assume that $0 \le g(x) \le f(x)$ for all $x \in [a, \infty)$.

1. If $\int_a^\infty f(x)\,dx$ converges then $\int_a^\infty g(x)\,dx$ also converges.

1. If $\int_a^\infty g(x)\,dx$ diverges then $\int_a^\infty f(x)\,dx$ also diverges.

Remark. Similar statements can be stated for improper integrals defined on intervals $(-\infty, b]$ and $[a, b]$.

Remark. 1. If $x \ge 1$ then $\frac{1}{x} \le \frac{1}{\sqrt{x}}$. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges then $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ also diverges.

2. If $0 < x \le 1$ then $\frac{1}{x} \le \frac{1}{x^2}$. Since $\int_0^1 \frac{1}{x}\,dx$ diverges then $\int_0^1 \frac{1}{x^2}\,dx$ also diverges.

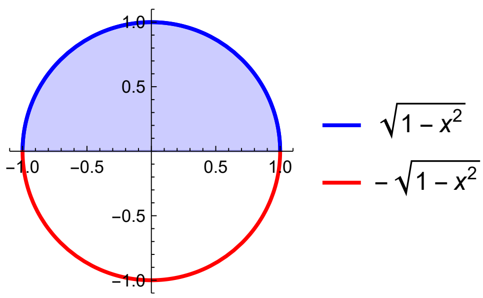
Applications

Area

Example. Calculate the area of the unit circle.

Solution. The equation of the circle with radius $r = 1$ centered at the origin is

$$x^2 + y^2 = 1 \implies y^2 = 1 - x^2 \implies y = \pm\sqrt{1 - x^2}$$



The area of the unit circle is $A = 2\int_{-1}^{1} \sqrt{1 - x^2}\,dx$

Substitution: $x = x(t) = \sin t \implies t = \arcsin x$

$$x'(t) = \frac{dx}{dt} = \cos t \implies dx = \cos t\, dt$$

The bounds will change: $x_1 = -1 \implies t_1 = \arcsin(-1) = -\frac{\pi}{2}$

$$x_2 = 1 \implies t_2 = \arcsin 1 = \frac{\pi}{2}$$

$$\implies A = 2\int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{-\pi/2}^{\pi/2} 2\sqrt{1-(\sin t)^2}\,\cos t\, dt = 2\int_{-\pi/2}^{\pi/2} \cos t \cdot \cos t\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \mathbf{2\cos^2 t}\, dt = \int_{-\pi/2}^{\pi/2} \mathbf{(1+\cos 2t)}\, dt = \left[t + \frac{\sin 2t}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= \left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2}\right) = \left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} + 0\right) = \pi$$

Arc length

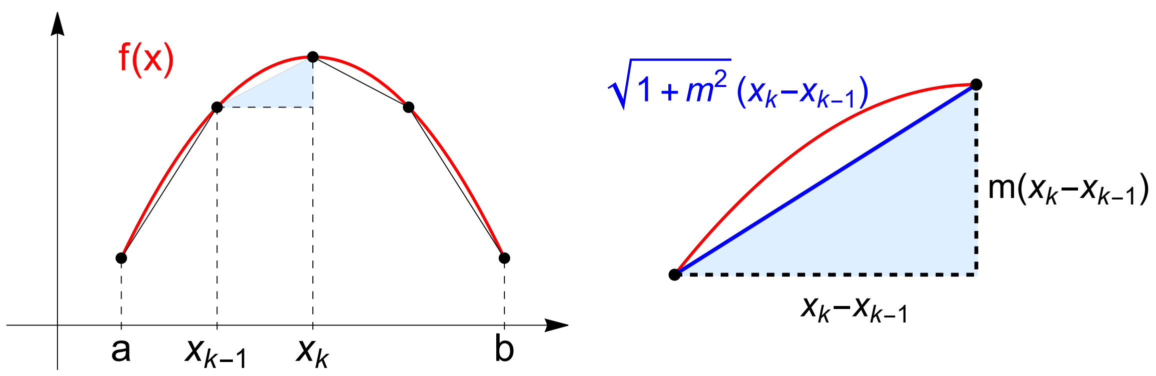
Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuously differentiable. Then the arc length of the graph of f is $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$.

Remark. Let $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a partition. If f is differentiable then by Lagrange's mean value theorem there exists $c_k \in (x_{k-1}, x_k)$ such that $m = f'(c_k)$, where m is the slope of the secant line connecting the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$.

So the arc length can be approximated by the sum $\sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2}\,(x_k - x_{k-1})$, which is the Riemann sum of the function $\sqrt{1 + (f'(x))^2}$.

If f is continuously differentiable then the arc length of the graph of f is

$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$.



Example. Calculate the arc length of the unit circle.

Solution. Let $f(x) = \sqrt{1-x^2}$ if $x \in [-1, 1]$.

$$f'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \frac{x}{\sqrt{1-x^2}}$$

$$\Longrightarrow \sqrt{1+(f'(x))^2} = \sqrt{1+\frac{x^2}{1-x^2}} = \sqrt{\frac{1}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

The arc length of the unit circle is

$$L = 2\int_{-1}^{1}\sqrt{1+(f'(x))^2}\,dx = 2\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx = 2\lim_{a\to-1+}\lim_{b\to 1-}\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx =$$

$$= 2\lim_{a\to-1+}\lim_{b\to 1-}[\arcsin x]_a^b = 2\lim_{a\to-1+}\lim_{b\to 1-}(\arcsin b - \arcsin a) =$$

$$= 2(\arcsin 1 - \arcsin(-1)) = 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2\pi$$

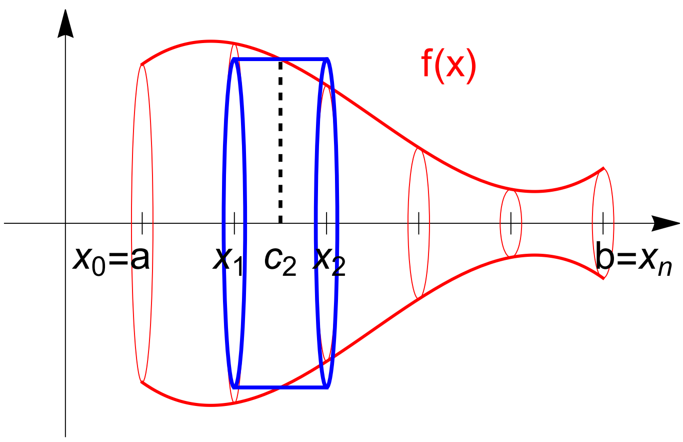
Volume of solids of revolutions

Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and nonnegative and the graph of f is rotated about the x axis. Then the volume of this solid of revolution is $V = \pi\int_a^b f^2(x)\,dx$.

Remark. If $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a partition then the volume can be approximated by the sum $\sum_{k=1}^{n}(x_k - x_{k-1})\,\pi f^2(c_k)$ where $c_k \in [x_{k-1}, x_k]$ is arbitrary.

(Geometrically it means that the volume can be approximated by the sum of volumes of cylinders.)

This is the Riemann sum of the function $\pi f^2(x)$, so if f is continuous then the volume is $V = \pi\int_a^b f^2(x)\,dx$.



Surface area of solids of revolutions

Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuously differentiable and nonnegative and the graph of f is rotated about the x axis. Then the surface area of this solid of revolution is

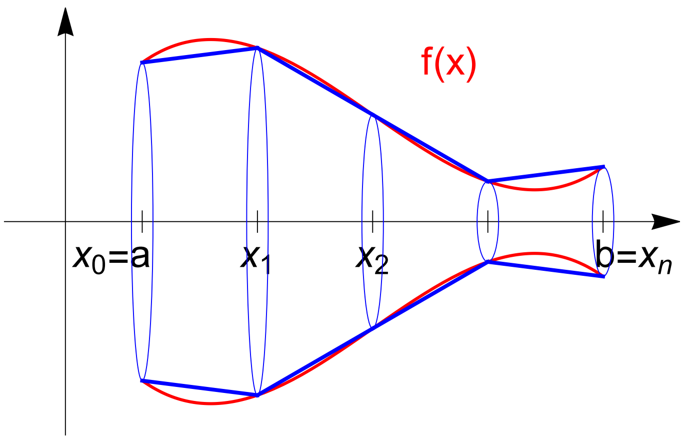
$$A = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

Remark. If $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a partition then the surface area of the solid of revolution can be approximated by the sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k)) \sqrt{1 + (f'(c_k))^2}\,(x_k - x_{k-1})$$

where $c_k \in [x_{k-1}, x_k]$ exists by the Lagrange intermediate value theorem if f is differentiable. (Geometrically it means that the surface area can be approximated by the sum of lateral surfaces of truncated cones.)

If f is continuously differentiable then $f(x_{k-1}) + f(x_k) \approx 2 f(c_k)$, so the above sum will be the Riemann sum of the function $2\pi f(x) \sqrt{1 + (f'(x))^2}$. Therefore if f is continuously differentiable then the surface area is $A = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}$ dx.



Exercise

Let $f(x) = \sqrt{r^2 - x^2}$, $-r \le x \le r$. Rotating the graph of f about the x axis, we get a sphere with radius r. Calculate the volume and surface area of the sphere.

Solution: 1. The volume can be calculated as $V = \pi \int_a^b f^2(x)\,dx$

The integrand is $(f(x))^2 = r^2 - x^2$

The volume is $V = \pi \int_{-r}^{r} (r^2 - x^2)\,dx = \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} =$

$= \pi\left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right) = \frac{4 r^3 \pi}{3}$

2. The surface are can be calculated as $A = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}$ dx

The derivative of f is $f'(x) = \left((r^2 - x^2)^{\frac{1}{2}}\right)' = \frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{r^2 - x^2}}$

$\Longrightarrow 1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$

The integrand is $f(x) \sqrt{1 + (f'(x))^2} = \sqrt{r^2 - x^2} \cdot \sqrt{\frac{r^2}{r^2 - x^2}} = r$

The surface area is $A = 2\pi \int_{-r}^{r} r\,dx = 2\pi \cdot [r\,x]_{-r}^{r} = 2\pi(r^2 - (-r^2)) = 4 r^2 \pi$

Integral test

Theorem. Assume that $f : [1, \infty) \longrightarrow \mathbb{R}$ be a positive valued, monotonically decreasing function and let $f(k) = a_k > 0$.

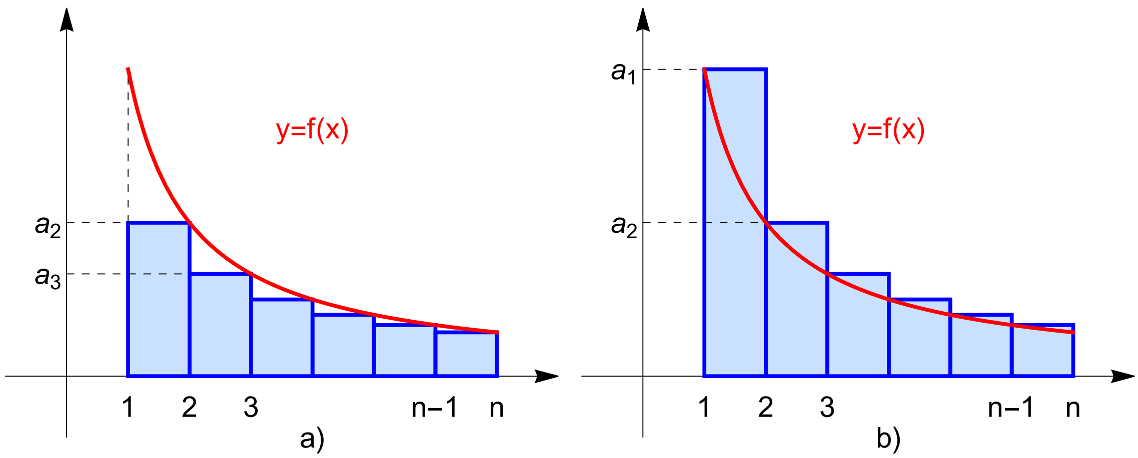
1. If $\int_1^\infty f(x)\,dx$ is convergent $\Longrightarrow$ $\sum_{k=1}^\infty a_k$ is convergent
2. If $\int_1^\infty f(x)\,dx$ is divergent $\Longrightarrow$ $\sum_{k=1}^\infty a_k$ is divergent

Remark. The equivalence is also true, that is, the integral $\int_1^\infty f(x)\,dx$ and the series $\sum_{k=1}^\infty a_k$ are both convergent or both divergent.

Proof. 1. Consider Figure a). Since the sum of the areas of the inscribed rectangles is less than or equal to the area under the graph of f then

$$a_2 + a_3 + \ldots + a_n \le \int_1^n f(x)\,dx \le \lim_{n\to\infty} \int_1^n f(x)\,dx = \int_1^\infty f(x)\,dx \in \mathbb{R}.$$

Since $a_k > 0$ and $\sum_{k=2}^n a_k$ is bounded $\Longrightarrow$ $\sum_{k=2}^\infty a_k$ is convergent $\Longrightarrow$ $\sum_{k=1}^\infty a_k$ is convergent.



2. Consider Figure b). Since the sum of the areas of the circumscribed rectangles is greater than or equal to the area under the graph of f then

$$\int_1^n f(x)\,dx \le a_1 + a_2 + \ldots + a_{n-1} = s_{n-1}$$

Since $\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty \Longrightarrow \lim_{n\to\infty} s_{n-1} = \infty \Longrightarrow \sum_{k=1}^\infty a_k$ is divergent.

Error estimation

Theorem: Let $f : [1, \infty) \longrightarrow \mathbb{R}$ be a positive valued, monotonically decreasing function, let $f(k) = a_k > 0$ and suppose that $\int_1^\infty f(x)\,dx$ is convergent. Let $s_n = \sum_{k=1}^{n} a_k$ and $s = \sum_{k=1}^{\infty} a_k$.

Then the error for the approximation $s \approx s_n$ is

$0 < E = s - s_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^\infty f(x)\,dx.$

Proof: Since $a_{n+1} + a_{n+2} + \ldots + a_m \leq \int_n^m f(x)\,dx$ then

$$0 < E = s - s_n = \lim_{m\to\infty} \sum_{k=n+1}^{m} a_k \leq \lim_{m\to\infty} \int_n^m f(x)\,dx = \int_n^\infty f(x)\,dx.$$

The convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$

Theorem: $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent if $\alpha > 1$ and divergent otherwise.

Proof: If $\alpha < 0$ then $\lim_{n\to\infty} \frac{1}{n^\alpha} = \lim_{n\to\infty} n^{-\alpha} = \lim_{n\to\infty} n^{|\alpha|} = \infty$ and

if $\alpha = 0$ then $\lim_{n\to\infty} \frac{1}{n^\alpha} = 1$, so in these cases the series is divergent by the nth term test.

If $\alpha > 0$ then let $f(x) = \frac{1}{x^\alpha}$, $x \geq 1$. This function is positive valued, monotonic decreasing and $f(n) = \frac{1}{n^\alpha} = a_n > 0$.

Thus, we can apply the integral test to investigate the convergence of $\sum_{n=1}^{\infty} a_n$.

We already proved that $\int_1^\infty \frac{1}{x^\alpha}\,dx$ is convergent if $\alpha > 1$, therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is also convergent if $\alpha > 1$.

The improper integral is divergent if $0 < \alpha \leq 1$, so in this case the series is also divergent.

Examples

Exercise. Decide whether the following series are convergent or divergent.

a) $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$ b) $\sum_{n=3}^{\infty} \frac{1}{n (\ln n)^2}$

Solution.

a) Let $f(x) = \frac{1}{x \ln x}$, $x > 3$

Then f is positive valued and monotonically decreasing on the interval $[3, \infty)$

and $f(n) = \dfrac{1}{n \ln n} > 0 \quad \Longrightarrow$ the integral test can be applied:

$$\int_3^\infty \frac{1}{x \ln x}\, dx = \lim_{A \to \infty} \int_3^A \frac{\frac{1}{x}}{\ln x}\, dx = \lim_{A \to \infty} [\ln(\ln x)]_3^A = \lim_{A \to \infty} (\ln(\ln A) - \ln(\ln 3)) = \infty$$

Since the improper integral is divergent then the series $\sum_{n=3}^{\infty} \dfrac{1}{n \ln n}$ is also divergent

by the integral test.

b) Let $f(x) = \dfrac{1}{x (\ln x)^2}, \quad x > 3$

Then f is positive valued and monotonically decreasing on the interval $[3, \infty)$

and $f(n) = \dfrac{1}{n (\ln n)^2} > 0 \quad \Longrightarrow$ the integral test can be applied:

$$\int_3^\infty \frac{1}{x (\ln x)^2}\, dx = \lim_{A \to \infty} \int_3^A \frac{1}{x} (\ln x)^{-2}\, dx = \lim_{A \to \infty} \left[-\frac{1}{\ln x}\right]_3^A = \lim_{A \to \infty} \left(-\frac{1}{\ln A} + \frac{1}{\ln 3}\right) = 0 + \frac{1}{\ln 3} = \frac{1}{\ln 3}$$

Since the improper integral is convergent then the series $\sum_{n=3}^{\infty} \dfrac{1}{n (\ln n)^2}$ is also convergent

by the integral test.