

21st and 22nd lectures

Analyzing graphs of functions

Summary of the steps:

- 1) finding the domain of f
- 2) finding the zeros of f
- 3) parity, periodicity
- 4) limits at the endpoints of the intervals constituting the domain
- 5) investigation of f' \Rightarrow monotonicity, extreme values
- 6) investigation of f'' \Rightarrow convexity/concavity, inflection points
- 7) linear asymptotes
- 8) plotting the graph of f , finding the range of f

Exercises with solutions: <https://math.bme.hu/~nagy/calculus1/functions.pdf>

Examples

$$1. f(x) = \frac{x}{x^3 + 1}$$

$$D_f = (-\infty, -1) \cup (-1, \infty); f(x) = 0 \iff x = 0;$$
$$\lim_{x \rightarrow \pm\infty} f(x) = 0, \quad \lim_{x \rightarrow -1+0} f(x) = -\infty, \quad \lim_{x \rightarrow -1-0} f(x) = +\infty$$

Monotonicity, local extremum:

$$f'(x) = \frac{1 - 2x^3}{(x^3 + 1)^2} = 0 \iff x = \frac{1}{\sqrt[3]{2}} \approx 0.79$$

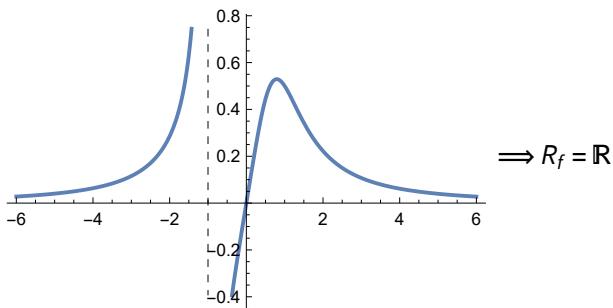
x	$x < -1$	$-1 < x < \frac{1}{\sqrt[3]{2}}$	$x = \frac{1}{\sqrt[3]{2}}$	$x > \frac{1}{\sqrt[3]{2}}$
f'	+	+	0	-
f	\nearrow	\nearrow	$\max: \frac{\sqrt[3]{4}}{3} \approx 0.53$	\searrow

Convexity / concavity, inflection points:

$$f''(x) = \frac{6x^2(x^3 - 2)}{(x^3 + 1)^3} = 0 \iff x = 0 \text{ or } x = \sqrt[3]{2} \approx 1.26$$

x	$x < -1$	$-1 < x < 0$	$x = 0$	$0 < x < \sqrt[3]{2}$	$x = \sqrt[3]{2}$	$x > \sqrt[3]{2}$
f''	+	-	0	-	0	+
f	\cup	\cap		\cap	$\text{infl: } \frac{\sqrt[3]{2}}{3} \approx 0.42$	\cup

The graph of f :



2. $f(x) = 2 \sin x + \sin 2x$

$D_f = \mathbb{R}; f$ is odd;

f is periodic with period $2\pi \Rightarrow$ it may be assumed that $0 \leq x \leq 2\pi$;

\Rightarrow on this interval $f(x) = 0 \Leftrightarrow x = 0$ or $x = \pi$ or $x = 2\pi$

Monotonicity, local extremum:

$$\begin{aligned} f'(x) &= 2 \cos x + 2 \cos 2x = 2(\cos x + \cos^2 x - (1 - \cos^2 x)) = \\ &= 2(2\cos^2 x + \cos x - 1) = 0 \Rightarrow (\cos x)_{1,2} = \frac{-1 \pm 3}{4} \Rightarrow \cos x = -1 \text{ or } \cos x = \frac{1}{2} \\ \Rightarrow x_1 &= \frac{\pi}{3}, x_2 = \pi, x_3 = \frac{5\pi}{3} \end{aligned}$$

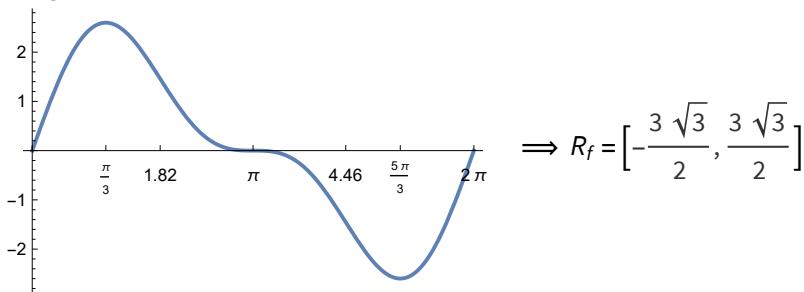
x	θ	$(0, \frac{\pi}{3})$	$\frac{\pi}{3}$	$(\frac{\pi}{3}, \pi)$	π	$(\pi, \frac{5\pi}{3})$	$\frac{5\pi}{3}$	$(\frac{5\pi}{3}, 2\pi)$	2π
f'	+	+	0	-	0	-	0	+	+
f		\nearrow	$\max: \frac{3\sqrt{3}}{2}$	\searrow		\searrow	$\min: -\frac{3\sqrt{3}}{2}$	\nearrow	

Convexity / concavity, inflection points:

$$\begin{aligned} f''(x) &= -2 \sin x - 4 \sin 2x = -2 \sin x - 8 \sin x \cos x = \\ &= -2 \sin x(1 + 4 \cos x) = 0 \Rightarrow \sin x = 0 \text{ or } \cos x = -\frac{1}{4} \\ \Rightarrow x_1 &= 0, x_2 = \pi, x_3 = 2\pi, x_4 = \arccos\left(-\frac{1}{4}\right) \approx 1.82, x_5 = 2\pi - \arccos\left(-\frac{1}{4}\right) \approx 4.46 \end{aligned}$$

x	θ	$(0, 1.82)$	1.82	$(1.82, \pi)$	π	$(\pi, 4.46)$	4.46	$(4.46, 2\pi)$	2π
f''	0	-	0	+	0	-	0	+	0
f	infl:0	\cap	infl:0 $\frac{3\sqrt{15}}{8}$	\cup	infl:0	\cap	infl:0 $-\frac{3\sqrt{15}}{8}$	\cup	infl:0

The graph of f :



Implicitly given curve

Example. The curve $y = y(x)$ is given by the following implicit equation:

$$x \sinh x - y \cosh y = 0$$

Study the properties of this curve in a neighbourhood of $(0, 0)$.

Solution. The point $(0, 0)$ is on the curve: $y(0) = 0$.

1) The first derivative of $x \sinh x - y(x) \cosh y(x) = 0$ with respect to x :

$$\sinh x + x \cosh x - y'(x) \cosh y(x) - y(x) y'(x) \sinh y(x) = 0$$

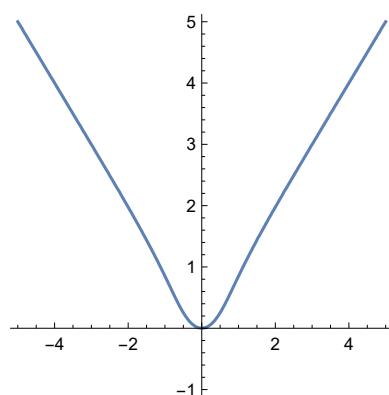
$$\text{If } x = 0, y = 0 \implies 0 + 0 \cdot 1 - y'(0) \cdot 1 - 0 \cdot y'(0) \cdot 0 = 0 \implies y'(0) = 0$$

2) The second derivative with respect to x :

$$\begin{aligned} & \cosh x + \cosh x + x \sinh x - y''(x) \cosh y(x) - y'(x) y'(x) \sinh y(x) \\ & - y'(x) y'(x) \sinh y(x) - y(x) y''(x) \sinh y(x) - y(x) y'(x) y'(x) \cosh y(x) = 0 \end{aligned}$$

$$\text{If } x = 0, y = 0 \implies 1 + 1 + 0 - y''(0) - 0 - 0 - 0 = 0 \implies y''(0) = 2$$

Since $y'(0) = 0$ and $y''(0) = 2 > 0$ then the curve $y = y(x)$ has local minimum at $x = 0$ and it is convex in some neighbourhood of $x = 0$.



Taylor polynomial

Definition. Let f be at least n times differentiable at x_0 . Then the polynomial

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is the **n th Taylor polynomial of f at x_0** . (If $x_0 = 0$: Maclaurin polynomial.)

Example. Let $f(x) = \sin x$ and $x_0 = 0$.

$$f(x) = \sin x \implies f(0) = 0 \implies T_0(x) = 0$$

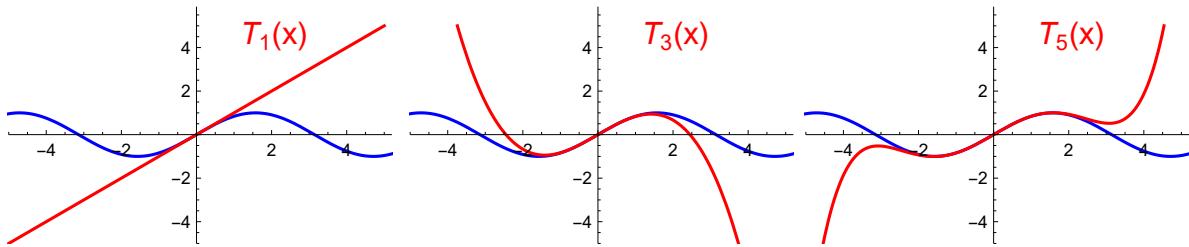
$$f'(x) = \cos x \quad f'(0) = 1 \quad T_1(x) = 0 + 1 \cdot x = x$$

$$f''(x) = -\sin x \quad f''(0) = 0 \quad T_2(x) = 0 + 1 \cdot x + \frac{0}{2!} x^2 = x$$

$$f'''(x) = -\cos x \quad f'''(0) = -1 \quad T_3(x) = 0 + 1 \cdot x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 = x - \frac{x^3}{6}$$

$$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0 \quad T_4(x) = 0 + 1 \cdot x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 = x - \frac{x^3}{6}$$

$$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1 \quad T_5(x) = 0 + 1 \cdot x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 = x - \frac{x^3}{6} + \frac{x^5}{120}$$



Theorem (Uniqueness of the Taylor polynomial).

Assume that f is at least n times differentiable at x_0 . Then $T_n(x)$ is the unique polynomial of degree at most n for which $T_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \dots, n$.

Proof. Express $T_n(x)$ as powers of $(x - x_0)$:

$$\begin{aligned} T_n(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 + \dots + a_n(x - x_0)^n \\ \implies T_n(x_0) &= a_0 = f(x_0) \end{aligned}$$

$$\begin{aligned} T_n'(x) &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^3 + \dots + na_n(x - x_0)^{n-1} \\ \implies T_n'(x_0) &= a_1 = f'(x_0) \end{aligned}$$

$$\begin{aligned} T_n''(x) &= 2a_2 + 3 \cdot 2 \cdot a_3(x - x_0) + 4 \cdot 3 \cdot a_4(x - x_0)^2 + \dots + n(n-1)a_n(x - x_0)^{n-2} \\ \implies T_n''(x_0) &= 2a_2 = f''(x_0) \end{aligned}$$

$$\begin{aligned} T_n'''(x) &= 3! \cdot a_3 + \dots + n(n-1)(n-2)a_n(x - x_0)^{n-3} \\ \implies T_n'''(x_0) &= 3!a_3 = f'''(x_0) \end{aligned}$$

Repeating the differentiation, we get that

$$T_n^{(k)}(x_0) = k!a_k = f^{(k)}(x_0), \dots, T_n^{(n)}(x_0) = n!a_n = f^{(n)}(x_0) \implies a_k = \frac{f^{(k)}(x_0)}{k!}, \quad k = 0, 1, \dots, n.$$

Theorem. If f is n times differentiable at x_0 then $\lim_{x \rightarrow x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} = 0$.

$$\text{Proof. } \lim_{x \rightarrow x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} \stackrel{L'H}{=} \lim_{x \rightarrow x_0} \frac{f'(x) - T_n'(x)}{n(x - x_0)^{n-1}} \stackrel{L'H}{=} \lim_{x \rightarrow x_0} \frac{f''(x) - T_n''(x)}{n(n-1)(x - x_0)^{n-2}} \stackrel{L'H}{=} \dots \stackrel{L'H}{=} \lim_{x \rightarrow x_0} \frac{f^{(n)}(x) - T_n^{(n)}(x)}{n!(x - x_0)^{n-n}} = \frac{0}{n!} = 0$$

Except the last one, the fractions are of the form $\frac{0}{0}$ so the L'Hospital's rule can be applied.

Taylor's theorem

Theorem (Taylor's theorem). Assume that f is at least $(n+1)$ times differentiable on the interval $(x_0 - \delta, x_0 + \delta)$ and $x \in (x_0 - \delta, x_0 + \delta)$. Then there exists a number ξ between x and x_0 (that is, $x_0 < \xi < x$ or $x < \xi < x_0$) such that

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}.$$

This expression is called the **Lagrange form of the remainder term**.

Proof. Assume that $x_0 < x$ and $y \in [x_0, x]$ (x is fixed).

Define the following function:

$$F(y) = f(y) + f'(y)(x-y) + \frac{f''(y)}{2!}(x-y)^2 + \dots + \frac{f^{(n)}(y)}{n!}(x-y)^n$$

Differentiating F with respect to y , we get a telescoping sum:

$$\begin{aligned} F'(y) &= f'(y) + (f''(y)(x-y) - f'(y)) + \left(\frac{f'''(y)}{2!}(x-y)^2 - \frac{f''(y)}{2!} \cdot 2 \cdot (x-y) \right) + \dots + \\ &\quad + \left(\frac{f^{(n+1)}(y)}{n!}(x-y)^n - \frac{f^{(n)}(y)}{n!} \cdot n \cdot (x-y)^{n-1} \right) = \frac{f^{(n+1)}(y)}{n!}(x-y)^n \end{aligned}$$

$$\Rightarrow F(x) = f(x)$$

$$\text{Let } G(y) = (x-y)^{n+1} \Rightarrow G(x) = 0$$

$$F(x_0) = T_n(x_0)$$

$$G(x_0) = (x-x_0)^{n+1}$$

$$F'(y) = \frac{f^{(n+1)}(y)}{n!}(x-y)^n$$

$$G'(y) = -(n+1)(x-y)^n$$

Both F and G are continuous on $[x_0, x]$, differentiable on (x_0, x) and $G'(y) \neq 0 \forall y \in (x_0, x)$, so by Cauchy's intermediate value theorem there exists $\xi \in (x_0, x)$ such that

$$\frac{f(x) - T_n(x_0)}{0 - (x-x_0)^{n+1}} = \frac{F(x) - F(x_0)}{G(x) - G(x_0)} = \frac{F'(\xi)}{G'(\xi)} = \frac{\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n}{\frac{-f^{(n+1)}(\xi)}{(n+1)!}(-n-1)(x-\xi)^n} = -\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\Rightarrow f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

Remark. If $n = 0$ then we obtain Lagrange's mean value theorem:

$$f(x) = T_0(x) + R_0(x) = f(x_0) + f'(\xi)(x-x_0)$$

Taylor series

Definition. If f is infinitely many times differentiable at x_0 then the power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

is called the **Taylor series** of f at x_0 .

Definition. The functions $g_0, g_1, \dots, g_n, \dots$ are **uniformly bounded** on the set $D = \bigcap D_{g_n}$ if

$$\exists K \in \mathbb{R}: |g_n(x)| \leq K \text{ if } x \in D, n \in \mathbb{N},$$

that is, the functions have a common bound K on the set D .

Example. The functions $\sin(x), 2\sin(2x), 3\sin(3x), \dots, n\sin(nx), \dots$ are each bounded but not uniformly bounded altogether.

The functions $\sin(x), \sin(2x), \sin(3x), \dots, \sin(nx), \dots$ are uniformly bounded, $K = 1$ is a suitable common bound.

Theorem. If f is infinitely many times differentiable on $(x_0 - R, x_0 + R)$ (R is the radius of convergence of T) and the functions $f, f', f'', f''', \dots, f^{(n)}, \dots$ are uniformly bounded on this interval then $f(x) = T(x)$ for all $x \in (x_0 - R, x_0 + R)$.

Proof. We have seen that $f(x) = T_n(x) + R_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$.

Using uniform boundedness and the limit $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$, we can give an upper estimation

for the remainder term:

$$|f(x) - T_n(x)| = |R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} \right| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} |x - x_0|^{n+1} \leq K \cdot \frac{R^{n+1}}{(n+1)!} \xrightarrow{n \rightarrow \infty} 0$$

Therefore $T_n(x) \rightarrow f(x)$ and thus $f(x) = T(x)$.

Remark. Using this theorem, it can be shown that many elementary functions are equal to their power series.

$$\textbf{Theorem. (1)} e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots, x \in \mathbb{R}$$

$$\textbf{(2)} \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots, x \in \mathbb{R}$$

$$\textbf{(3)} \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots, x \in \mathbb{R}$$

$$\textbf{(4)} \sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots, x \in \mathbb{R}$$

$$\textbf{(5)} \cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots, x \in \mathbb{R}$$

In each case, the interval of convergence is the set of real numbers.

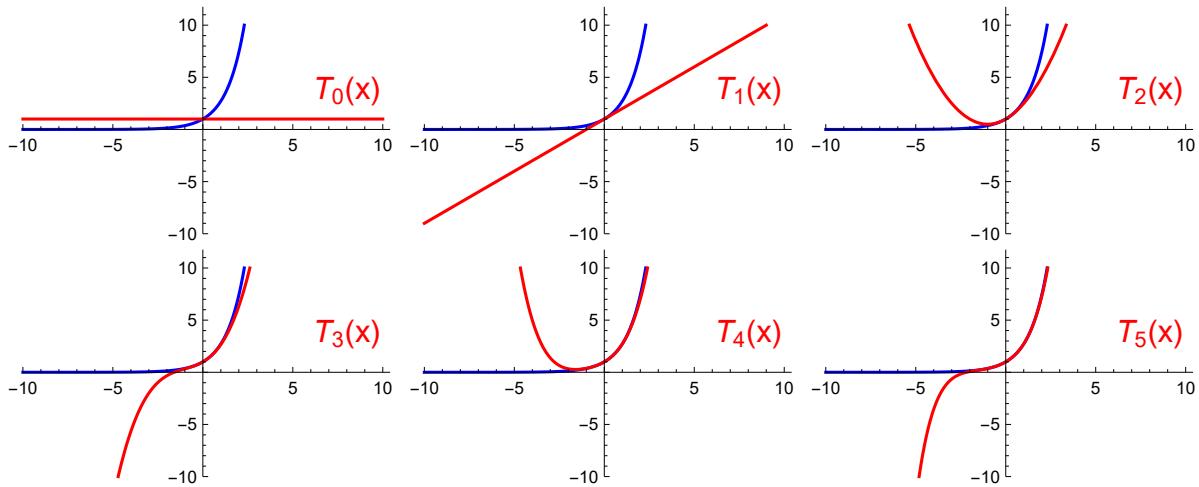
Proof. (1) Let $f(x) = e^x$ and $x_0 = 0 \implies f^{(k)}(x) = e^x \forall k \in \mathbb{N} \implies f^{(k)}(0) = 1$.

By the ratio test the radius of convergence is $\lim_{n \rightarrow \infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty$

\implies the interval of convergence is \mathbb{R} .

The derivatives are not bounded on \mathbb{R} , however, they are uniformly bounded on any closed interval $[a, b]$ and $|f^{(k)}(x)| \leq e^b = K \implies f(x) = T(x)$ if $x \in [a, b]$.

Since this equality holds for **any** closed interval $[a, b]$ then $f(x) = T(x) \forall x \in \mathbb{R}$ also holds.

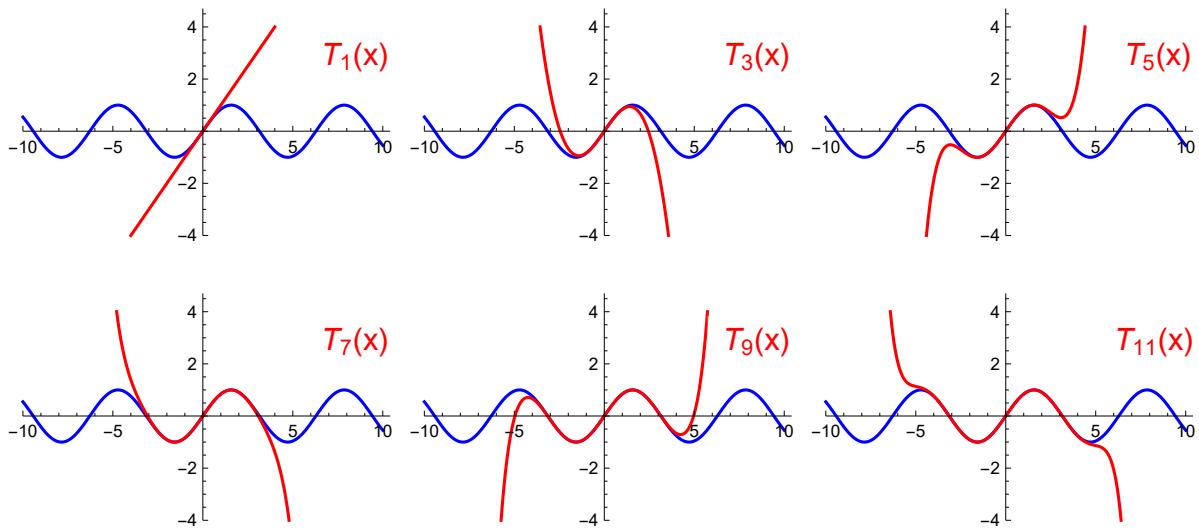


(2) Let $f(x) = \sin x$ and $x_0 = 0$.

$$\begin{aligned} f(x) &= \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = f(x) = \sin x \\ f(0) &= 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^{(4)}(0) = f(0) = 0 \end{aligned}$$

From here it is repeated periodically. The derivatives are uniformly bounded on \mathbb{R}

$$\begin{aligned} \implies \cos x &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots + \frac{f^{(k)}(0)}{k!} x^k + \dots = \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \end{aligned}$$

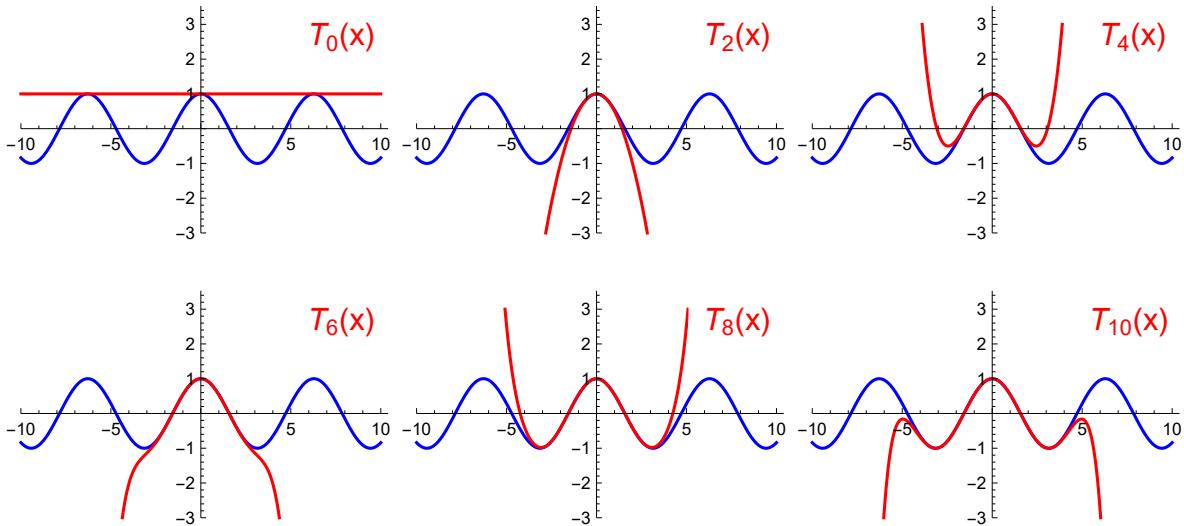


(3) Let $f(x) = \cos x$ and $x_0 = 0$.

$$f(x) = \cos x, f'(x) = -\sin x, f''(x) = -\cos x, f'''(x) = \sin x, f^{(4)}(x) = f(x) = \cos x \\ f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0, f^{(4)}(0) = f(0) = 1$$

From here it is repeated periodically. The derivatives are uniformly bounded on \mathbb{R}

$$\Rightarrow \cos x = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots + \frac{f^{(k)}(0)}{k!} x^k + \dots = \\ = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$



(4), (5)

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (x \in \mathbb{R}) \quad \Rightarrow$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n - (-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (x \in \mathbb{R}) \quad \cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n + (-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

Remark. The Taylor series of $f(x) = e^x$ at x_0 is $e^x = e^{x_0} e^{x-x_0} = \sum_{n=0}^{\infty} \frac{e^{x_0}}{n!} (x-x_0)^n$

Examples. (1) The Taylor series of $f(x) = \sin x \cdot \cos x$ at $x_0 = 0$ is

$$f(x) = \sin x \cdot \cos x = \frac{1}{2} \sin 2x = \frac{1}{2} \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots \right) \quad (x \in \mathbb{R})$$

$$(2) \text{ The Taylor series of } f(x) = e^{-x^2} \text{ at } x_0 = 0 \text{ is } e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots \quad (x \in \mathbb{R})$$

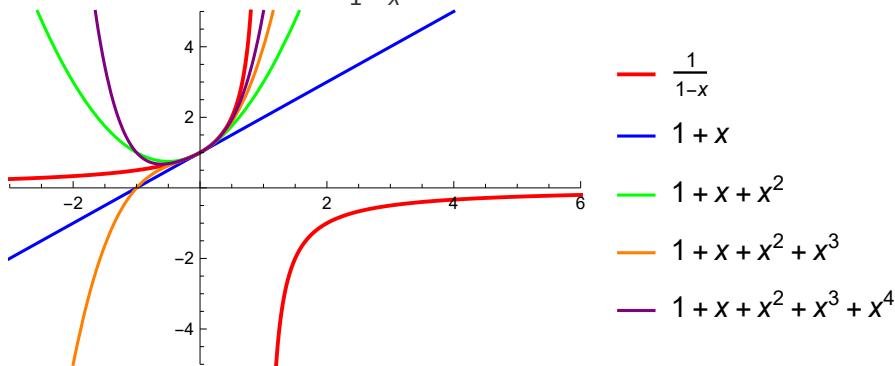
Example. It is known that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$ (sum of a geometric series).

Find the Taylor series of $f(x) = \frac{1}{1-x}$ at $x_0 = 0$.

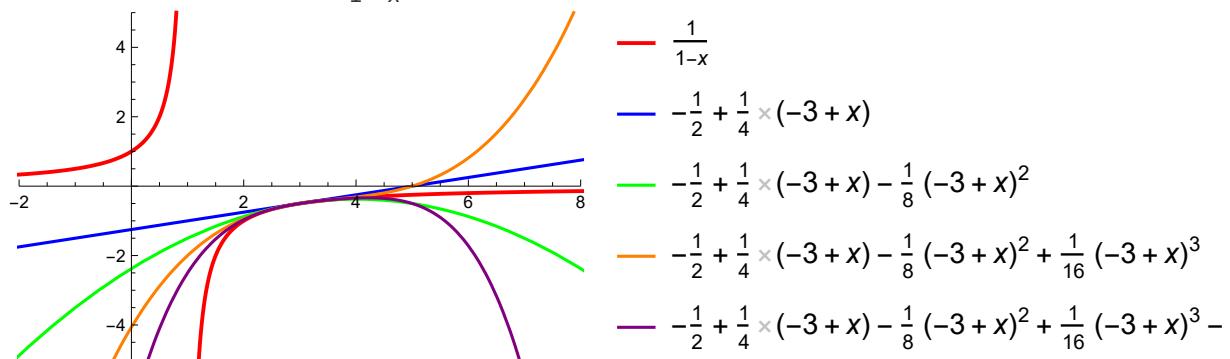
$$f'(x) = \frac{1}{(1-x)^2}, f''(x) = \frac{2}{(1-x)^3}, f'''(x) = \frac{3!}{(1-x)^4}, \dots, f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \Rightarrow f^{(n)}(0) = n!$$

$$\Rightarrow T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ if } |x| < 1.$$

Taylor polynomials of $f(x) = \frac{1}{1-x}$ with center $x_0 = 0$:



Taylor polynomials of $f(x) = \frac{1}{1-x}$ with center $x_0 = 3$:



Term by term differentiation and integration

Theorem. If $|x - x_0| < R$ where R is the radius of convergence of the

Taylor series $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, then $f'(x) = \sum_{n=0}^{\infty} n a_n(x - x_0)^{n-1}$

and if $f'(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, then $f(x) = f(x_0) + \sum_{n=0}^{\infty} a_n \frac{(x - x_0)^{n+1}}{n+1}$.

$$\text{Example. } (e^x)' = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right)' = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x$$

$$(\sin x)' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots\right)' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots = \cos x$$

$$\text{Statement. } \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ if } x \in [-1, 1].$$

Proof. $f(x) = \arctan x \implies$

$$f'(x) = (\arctan x)' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \dots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$f'(x)$ is the sum of a geometric series with ratio $q = -x^2$.

It is convergent $\iff |q| = |-x^2| = |x|^2 < 1 \implies R = 1$

The interval of convergence for f' is $(-1, 1)$.

$$\Rightarrow f(x) = \arctan x = f(0) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots$$

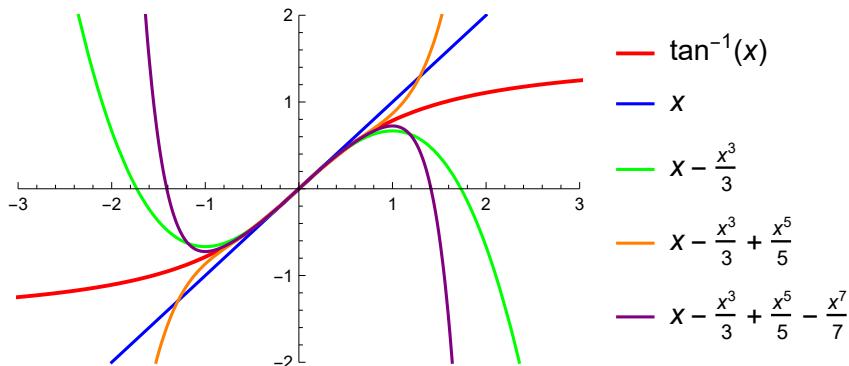
The radius of convergence doesn't change but the endpoints of the interval of convergence can change. Here both endpoints change.

If $x = 1$ then

$$f(1) = \arctan 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots, \text{ this is a Leibniz type series, so it is convergent.}$$

If $x = -1$ then $f(-1) = -f(1)$, so the series is also convergent.

The interval of convergence for f is $[-1, 1]$.



Statement. $\ln(1+x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ if } x \in [-1, 1].$

Proof. $f(x) = \ln(1+x) \Rightarrow$

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$f'(x)$ is the sum a geometric series with ratio $q = -x$.

It is convergent $\Leftrightarrow |q| = |-x| = |x| < 1 \Rightarrow R = 1$.

The interval of convergent for f' is $(-1, 1)$.

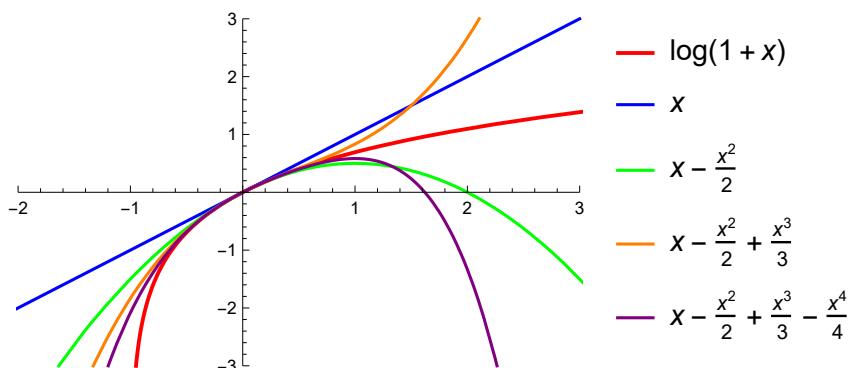
$$\Rightarrow f(x) = \ln(1+x) = f(0) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

The radius of convergence doesn't change but the endpoints of the interval of convergence can change. Here the right endpoint changes.

If $x = -1$ then $f(x)$ is not defined. If $x = 1$ then

$$f(1) = \ln 2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots, \text{ this is a Leibniz type series, so it is convergent.}$$

The interval of convergence for f is: $(-1, 1]$.



Binomial series

Example. Find the k th Taylor polynomial of the function $f(x) = (1+x)^\alpha$ (where $\alpha \in \mathbb{R}$) about $x_0 = 0$.

$$\begin{aligned} f(x) &= (1+x)^\alpha &\Rightarrow f(0) &= 1 \\ f'(x) &= \alpha(1+x)^{\alpha-1} &\Rightarrow f'(0) &= \alpha \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} &\Rightarrow f''(0) &= \alpha(\alpha-1) \\ &\dots \\ f^{(k)}(x) &= \alpha(\alpha-1)(\alpha-k+1)(1+x)^{\alpha-k} \Rightarrow f^{(k)}(0) &&= \alpha(\alpha-1)\dots(\alpha-k+1) \end{aligned}$$

$$\Rightarrow T_k(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 \dots + \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k.$$

Remark. If $\alpha = n \in \mathbb{N}$ then by the binomial theorem

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \text{ where } \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

Definition (Generalized binomial coefficient). If $\alpha \in \mathbb{R}$ and $k \in \mathbb{R}$ then

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \text{ and } \binom{\alpha}{0} := 1$$

Remark. The Taylor series of $f(x) = (1+x)^\alpha$ (where $\alpha \in \mathbb{R}$) is $T(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

It is called a **binomial series**.

Theorem. The radius of convergence of the binomial series $\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ is $R = 1$.

Proof.

$$\begin{aligned} a_k &= \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \\ a_{k+1} &= \binom{\alpha}{k+1} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)(\alpha-k)}{(k+1)!} = \binom{\alpha}{k} \frac{\alpha-k}{k+1} \end{aligned}$$

By the ratio test:

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\alpha-k}{k+1} \right| = 1 = \frac{1}{R} \Rightarrow R = 1$$

Theorem (Sum of a binomial series).

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \text{ for all } x \in (-1, 1), \alpha \in \mathbb{R}.$$

Proof. The derivatives of $f(x)$ are not uniformly bounded, so we prove the equality of

$$f(x) = (1+x)^\alpha \text{ and } T(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \text{ in the following way.}$$

We show that for all $x \in (-1, 1)$: $\left(\frac{T(x)}{f(x)} \right)' \equiv 0 \Rightarrow \frac{T(x)}{f(x)} \equiv \text{constant}$.

Since $f(0) = T(0) = 1$ then $\frac{T(x)}{f(x)} = \frac{T(0)}{f(0)} = 1 \Rightarrow T(x) = f(x)$ if $x \in (-1, 1)$.

The derivative is

$$\begin{aligned}\left(\frac{T(x)}{f(x)}\right)' &= \frac{T' \cdot f - T \cdot f'}{f^2} = \frac{T' \cdot (1+x)^\alpha - T \cdot \alpha(1+x)^{\alpha-1}}{f^2} = \\ &= \frac{(1+x)^{\alpha-1}}{(1+x)^{2\alpha}} ((1+x)T' - \alpha \cdot T)\end{aligned}$$

\Rightarrow it is enough to show that $(1+x)T' - \alpha \cdot T \equiv 0$.

For this we use the power series of T and T' .

$$\begin{aligned}T(x) &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \binom{\alpha}{k} x^k + \dots \\ \alpha T(x) &= \alpha + \alpha^2 x + \frac{\alpha^2(\alpha-1)}{2!} x^2 + \dots + \alpha \binom{\alpha}{k} x^k + \dots \\ T'(x) &= \alpha + \frac{\alpha(\alpha-1)}{1!} x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!} x^2 + \dots + k \binom{\alpha}{k} x^{k-1} + (\textcolor{red}{k+1}) \binom{\alpha}{k+1} x^k + \dots \\ x T'(x) &= \alpha x + \frac{\alpha(\alpha-1)}{1!} x^2 + \dots + \textcolor{blue}{k} \binom{\alpha}{k} x^k + \dots\end{aligned}$$

The above expression as a power series is

$$(1+x)T' - \alpha \cdot T = \sum_{k=0}^{\infty} \left((\textcolor{red}{k+1}) \binom{\alpha}{k+1} + \textcolor{blue}{k} \binom{\alpha}{k} - \alpha \binom{\alpha}{k} \right) x^k$$

where the coefficient of x^k for all $k \in \mathbb{N}$ is

$$(k+1) \binom{\alpha}{k+1} + k \binom{\alpha}{k} - \alpha \binom{\alpha}{k} = \binom{\alpha}{k} \left((k+1) \frac{\alpha-k}{k+1} + k - \alpha \right) = 0.$$

Thus we proved that for all $x \in (-1, 1)$

$$(1+x)T'(x) - \alpha \cdot T(x) \equiv 0 \Rightarrow \left(\frac{T}{f}\right)' \equiv 0 \Rightarrow \frac{T(x)}{f(x)} = \frac{T(0)}{f(0)} = 1$$

$$\Rightarrow T(x) = f(x) \text{ if } x \in (-1, 1).$$

Example. If $f(x) = \arcsin x$, then $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1+(-x^2))^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{2k}$ if $|x| < 1$

$$\Rightarrow \arcsin x = \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k x^{2k+1}}{2k+1} \text{ if } |x| < 1.$$

Remark. Taylor's formula \Rightarrow

$$(1+x)^\alpha = T_n(x) + R_n(x), \text{ where}$$

$$T_n(x) = \sum_{k=0}^n \binom{\alpha}{k} x^k \text{ and } R_n(x) = \binom{\alpha}{n+1} (1+\xi)^{\alpha-n-1} x^{n+1}, \text{ where } 0 < \xi < x \text{ or } x < \xi < 0.$$

Exercises

Exercise 1. Estimate the value of $\sqrt{2}$ by the Taylor polynomial of order 2 of $f(x) = \sqrt{1+x}$ at center $x_0 = 0$. Give an upper bound for the error for the error of the approximation.

Solution. The derivatives and the substitution values:

$$\begin{aligned}f(x) &= \sqrt{1+x} = (1+x)^{\frac{1}{2}}, \quad f(0) = 1 \\ f'(x) &= \frac{1}{2} (1+x)^{-\frac{1}{2}} = \frac{1}{2 \sqrt{1+x}}, \quad f'(0) = \frac{1}{2}\end{aligned}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} = -\frac{1}{4(1+x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} = \frac{3}{8(1+x)^{5/2}}$$

The Taylor polynomial of order 2:

$$T_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 = 1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2$$

$$f(x) \approx T_2(x), \text{ that is, } \sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{4 \cdot 2!}x^2$$

$$\text{If } x=1 \text{ then } \sqrt{2} \approx T_2(1) = 1 + \frac{1}{2} - \frac{1}{4 \cdot 2!} = 1.375$$

$$\text{Lagrange remainder term: } R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-x_0)^3, \text{ where } x_0=0, x=1, 0 < \xi < 1$$

The value of ξ is not known so we can only estimate the error.

$$|E| = |R_2(x)| = \left| \frac{3}{8(1+\xi)^{5/2}} \cdot \frac{1}{3!} (1-0)^3 \right| = \frac{1}{16(1+\xi)^{5/2}} < \frac{1}{16(1+0)^{5/2}} = \frac{1}{16} = 0.0625$$

Remark: $\sqrt{2} \approx 1.414213562 \dots$

The approximation is $\sqrt{2} \approx T_2(1) = 1.375$

$$1.4142 - 1.375 = 0.0392$$

$$\text{Remark: } f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$$

$$\Rightarrow T_2(x) = \sum_{k=0}^2 \binom{1/2}{k} x^k = 1 + \frac{1}{2}x + \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right)}{2!} x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$R_2(x) = \binom{1/2}{3} (1+\xi)^{\frac{1}{2}-3} x^3 = \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right)}{3!} \cdot \frac{1}{(1+\xi)^{\frac{5}{2}}} \cdot x^3 = \frac{3}{8 \cdot 3!} \cdot \frac{1}{(1+\xi)^{\frac{5}{2}}} \cdot x^3$$

Exercise 2. We estimate the value of $\ln(1.1)$ by the Taylor polynomial of order n of

$f(x) = \ln(1+x)$ at center $x_0 = 0$. Find n if the error for the approximation is less than 10^{-8} .

Solution. The first few derivatives are

$$f(x) = \ln(1+x), \quad f(0) = 0 \quad f'''(x) = \frac{2}{(1+x)^3}, \quad f'''(0) = 2 = 2!$$

$$f'(x) = \frac{1}{1+x}, \quad f'(0) = 1 = 0! \quad f^{(4)}(x) = -\frac{6}{(1+x)^4}, \quad f^{(4)}(0) = -6 = -3!$$

$$f''(x) = -\frac{1}{(1+x)^2}, \quad f''(0) = -1 = -1! \quad f^{(5)}(x) = \frac{24}{(1+x)^5}, \quad f^{(5)}(0) = 24 = 4! \text{ etc.}$$

$$\Rightarrow f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n} \Rightarrow f^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!$$

The Taylor polynomial at $x_0 = 0$:

$$T_n(x) = 0 + 1 \cdot (x - 0) - \frac{1}{2!} (x - 0)^2 + \frac{2}{3!} (x - 0)^3 + \frac{-6}{4!} (x - 0)^4 + \dots = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 +$$

$$\text{If } x = 0.1: \ln(1.1) \approx 0.1 - \frac{1}{2} 0.1^2 + \frac{1}{3} 0.1^3 - \frac{1}{4} 0.1^4 + \dots$$

Lagrange remainder term: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$, where $x_0 = 0$, $x = 0.1$, $0 < \xi < 0.1$.

Taylor's theorem: $f(x) = T_n(x) + R_n(x)$

$$\begin{aligned} \text{The error: } |E| &= |f(x) - T_n(x)| = |R_n(x)| = \left| \frac{(-1)^n \cdot n!}{(1+\xi)^{n+1} \cdot (n+1)!} (x - x_0)^{n+1} \right| = \\ &= \left| \frac{(-1)^n \cdot n!}{(1+\xi)^{n+1} \cdot (n+1)!} (0.1 - 0)^{n+1} \right| = \frac{1}{(n+1)(1+\xi)^{n+1}} \cdot 0.1^{n+1} < \frac{1}{(n+1)(1+0)^{n+1}} \cdot 0.1^{n+1} = \\ &= \frac{0.1^{n+1}}{(n+1)} < 10^{-8} \implies n \geq 7 \end{aligned}$$

Comparison of the numerical values:

$$\ln[11/10] := \mathbf{N}\left[\mathbf{Log}\left[\frac{11}{10}\right], 10\right]$$

$$\text{Out}[1]:= 0.09531017980$$

$$\ln[11/10] := \mathbf{N}\left[x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \frac{1}{5} x^5 - \frac{1}{6} x^6 + \frac{1}{7} x^7 / . \rightarrow \frac{1}{10}, 10\right]$$

$$\text{Out}[2]:= 0.09531018095$$

\implies the first 7 digits are accurate.