19th and 20th lectures

L'Hospital's rule

Remark. The following theorem can be applied for limits of the type
$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty-\infty,\ 0\cdot\infty,\ 1^{\infty},\ \infty^{0}$.

Theorem (L'Hospital's rule).
Assume that $a \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, I is a neighbourhood of a, the functions f and g are differentiable on $I \setminus \{a\}$ and $g(x) \neq 0$, $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$.
Assume moreover that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty.$$

If $\exists \lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = b \in \overline{\mathbb{R}}$ then $\exists \lim\limits_{x\to a} \frac{f(x)}{g(x)} = b$.

Remark. The theorem holds for right-hand and left-hand limits as well.

Proof. 1st case (for right-hand limit).

Assume that $a \in \mathbb{R}$, $\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a+} g(x) = 0$ and $\exists \lim\limits_{x\to a+} \frac{f'(x)}{g'(x)} = b \in \mathbb{R}$.

Extend the functions f and g such that $f(a) = g(a) = 0$ and let $x \in I$, $x > a$.
Then f and g are continuous on $[a, x]$ and differentiable on (a, x),
so by Cauchy's mean value theorem there exists $c \in (a, x)$ such that
$$\frac{f(x)}{g(x)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}.$$
Let (x_n) be a sequence such that $x_n \longrightarrow a$ and choose $c_n \in (a, x_n)$ for all n.
Then $c_n \longrightarrow a$ and $\frac{f(x_n)}{g(x_n)} = \frac{f'(c_n)}{g'(c_n)}$ for all $n \in \mathbb{N}$.
Therefore $\lim\limits_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \lim\limits_{n\to\infty} \frac{f'(c_n)}{g'(c_n)} = b$ and by the sequential criterion for the limit, $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = b$.

2nd case.

Assume that $a \in \mathbb{R}$, $\lim\limits_{x\to a} |f(x)| = \lim\limits_{x\to a} |g(x)| = \infty$ and $\exists \lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = b \in \overline{\mathbb{R}}$.

Let $A = \lim\limits_{x\to a} \frac{f(x)}{g(x)}$.

(1) If $A, b \in \mathbb{R}$, $A \neq 0$, $b \neq 0$:

$$\Longrightarrow A = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}} = \lim_{x\to a} \frac{-\frac{g'(x)}{g^2(x)}}{-\frac{f'(x)}{f^2(x)}} = \lim_{x\to a} \frac{f^2(x)\, g'(x)}{g^2(x)\, f'(x)} =$$

$$= \lim_{x\to a}\frac{f^2(x)}{g^2(x)}\cdot\lim_{x\to a}\frac{g'(x)}{f'(x)} = A^2\cdot\frac{1}{b} \Longrightarrow A = b$$

(2) If $A = 0$:

$$1+\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)+g(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)+g'(x)}{g'(x)} = 1+\lim_{x\to a}\frac{f'(x)}{g'(x)} \Longrightarrow A = b$$

(3) If $A = \lim_{x\to a}\frac{f(x)}{g(x)} = \begin{cases}+\infty\\-\infty\end{cases} \Longrightarrow \lim_{x\to a}\frac{g(x)}{f(x)} = \lim_{x\to a}\frac{g'(x)}{f'(x)} = \begin{cases}0+\\0-\end{cases} \Longrightarrow \lim_{x\to a}\frac{f'(x)}{g'(x)} = \begin{cases}+\infty\\-\infty\end{cases}$

3rd case.

Assume that $x\longrightarrow+\infty$ and let $t = \frac{1}{x}$. Then $t\longrightarrow 0+$ if $x\longrightarrow+\infty$.

$$\Longrightarrow \lim_{x\to+\infty}\frac{f(x)}{g(x)} = \lim_{t\to 0+}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t\to 0+}\frac{\frac{d}{dt}f\left(\frac{1}{t}\right)}{\frac{d}{dt}g\left(\frac{1}{t}\right)} = \lim_{t\to 0+}\frac{-\frac{f'\left(\frac{1}{t}\right)}{t^2}}{-\frac{g'\left(\frac{1}{t}\right)}{t^2}} = \lim_{t\to 0+}\frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = \lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$$

If $x\longrightarrow-\infty$ then $t\longrightarrow 0-$ and the proof is similar.

Local properties and the derivative

Definition. Assume that $x_0\in D_f$ and there exists $\delta>0$ such that for all $x,\ y\in D_f$, if $x_0-\delta<x<x_0<y<x_0+\delta$,

then $\begin{cases} f(x)\le f(x_0)\le f(y)\\ f(x)\ge f(x_0)\ge f(y)\\ f(x)<f(x_0)<f(y)\\ f(x)>f(x_0)>f(y)\end{cases}$. Then we say that f is $\begin{cases}\text{locally increasing}\\ \text{locally decreasing}\\ \text{strictly locally increasing}\\ \text{strictly locally decreasing}\end{cases}$ at x_0.

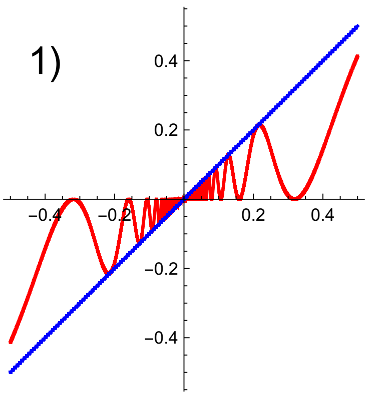
Remarks. (1) If f is monotonically increasing on (a,b), then f is locally increasing for all $x_0\in(a,b)$.

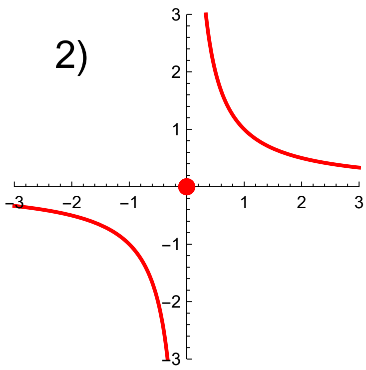
(2) If f is locally increasing **for all** $x_0\in(a,b)$, then f is monotonically increasing on (a,b).

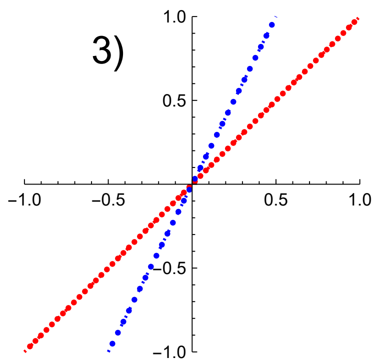
(3) However, if f is locally increasing at x_0 then it doesn't imply that there exists a neighbourhood $B(x_0,r)$ where f is monotonically increasing.

Examples. The following functions are locally increasing at $x_0=0$ but on any interval that contains 0, the functions are not monotonically increasing.

1. $f(x) = \begin{cases} x\sin^2\frac{1}{x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ **2.** $f(x) = \begin{cases}\frac{1}{x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ **3.** $f(x) = \begin{cases} x & \text{if } x\in\mathbb{Q}\\ 2x & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$







Theorem. Assume that f is differentiable at x_0.

(1) If f is locally increasing at x_0 then $f'(x_0) \geq 0$.

(2) If f is locally decreasing at x_0 then $f'(x_0) \leq 0$.

(3) If $f'(x_0) > 0$ then f is strictly locally increasing at x_0.

(4) If $f'(x_0) < 0$ then f is strictly locally decreasing at x_0.

Proof. (1) If f is locally increasing at x_0 then $\exists\, \delta > 0$ such that

$$0 < |x - x_0| < \delta \Longrightarrow \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

(If $x < x_0$ then $x - x_0 < 0$ and $f(x) - f(x_0) \leq 0$ and if $x > x_0$ then $x - x_0 > 0$ and $f(x) - f(x_0) \geq 0$.)

Since f is differentiable at x_0 then $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$.

(2) Similar to case (1).

(3) If $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} > 0$, then there exists $\delta > 0$ such that

if $0 < |x - x_0| < \delta$ then $\frac{f(x) - f(x_0)}{x - x_0} > 0$.

$$\Longrightarrow \text{ if } \begin{cases} x_0 < x < x_0 + \delta \\ x_0 - \delta < x < x_0 \end{cases} \text{ then } \begin{cases} f(x) > f(x_0) \\ f(x) < f(x_0) \end{cases}$$

$\Longrightarrow$ f is strictly locally increasing at x_0.

(3) Similar to case (4).

Remarks. Assume that f is differentiable at x_0.

(1) If f is strictly locally increasing at x_0 then it doesn't imply that $f'(x_0) > 0$.

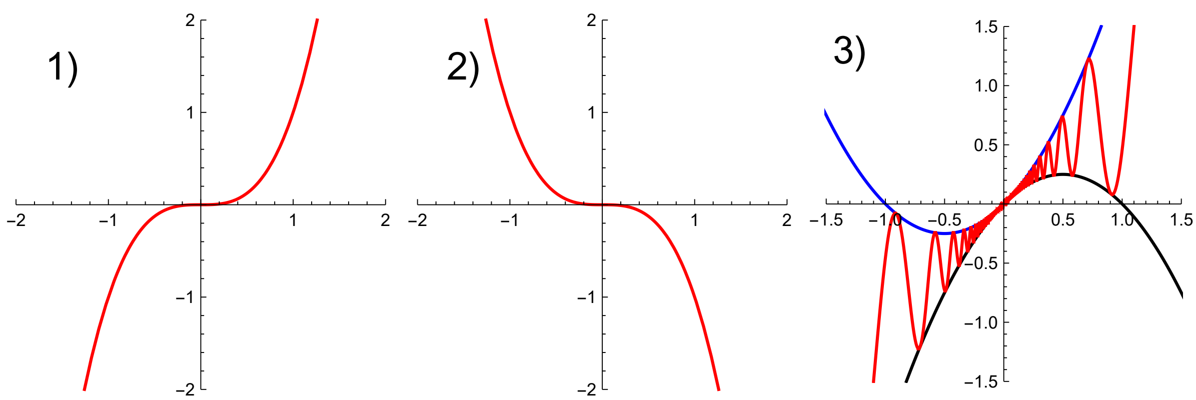
If f is strictly locally increasing at x_0 then $f'(x_0) \geq 0$, since $\exists\, \delta > 0$ such that

$0 < |x - x_0| < \delta \Longrightarrow \frac{f(x) - f(x_0)}{x - x_0} > 0$, but for the limit $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$.

For example $f(x) = x^3$ is strictly locally increasing at $x_0 = 0$, but $f'(0) = 3x^2 \,|_{x=0} = 0$.

(2) If $f'(x_0) \geq 0$ then it doesn't imply that f is locally increasing at x_0.

For example $f(x) = -x^3$ is not locally increasing at $x_0 = 0$, but $f'(0) = \ \geq 0$.



(3) If $f'(x_0) > 0$ then it doesn't imply that f is monotonically increasing on an interval containing x_0.

For example, let f be a function such that $x - x^2 \le f(x) \le x + x^2 \ \forall x \implies f(0) = 0$.

If $x > 0$ then $1 - x \le \frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} \le 1 + x$,

If $x < 0$ then $1 - x \ge \frac{f(x) - f(0)}{x - 0} \ge 1 + x$, so by the sandwich theorem

$f'(0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = 1 > 0$. For example, let $f(x) = \begin{cases} x + x^2 \sin\left(\frac{10}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

Darboux's theorem

Theorem. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable and $f'(a) < y < f'(b)$ or $f'(b) < y < f'(a)$.
Then there exists $c \in (a, b)$ such that $f'(c) = y$.

Remark. We say that f' has the intermediate value property of Darboux property.

Proof. 1) Let $g : [a, b] \longrightarrow \mathbb{R}$, $g(x) = f(x) - y \cdot x \implies g$ is differentiable and $g'(x) = f'(x) - y$.

2) Assume that $f'(a) < y < f'(b) \implies g'(a) = f'(a) - y < 0 < f'(b) - y < g'(b)$

3) g is differentiable, so it is continuous on $[a, b]$

$\implies$ by Weierstrass extreme value theorem it has a minimum and a maximum on $[a, b]$.

4) Since $\begin{cases} g'(a) < 0 \\ g'(b) > 0 \end{cases}$ then $\begin{cases} g \text{ is strictly locally decreasing at } a \\ g \text{ is strictly locally increasing at } b \end{cases}$

$\implies g$ does not have a minimum at a and b but on the open interval (a, b)

$\implies$ there exists $c \in (a, b)$ such that g has a local minimum at c

$\implies g'(c) = 0 = f'(c) - y \implies f'(c) = y$ for some $c \in (a, b)$.

Example. The sign function or signum function is defined as $\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \\ 1 & \text{if } x > 0 \end{cases}$

This function is not continuous at $x_0 = 0$, so there is no function $f : \mathbb{R} \longrightarrow \mathbb{R}$
for which $f'(x) = \operatorname{sgn} x$ on $\mathbb{R}$ (or on any interval that contains $x_0 = 0$).

Remark. From Darboux's theorem it follows that if f' is not continuous at a point then
f' cannot have a discontinuity of the first type at that point, so at least one of the one-sided limits doesn't exist or exists but is not finite
$\implies f'$ has an essential discontinuity at the given point.

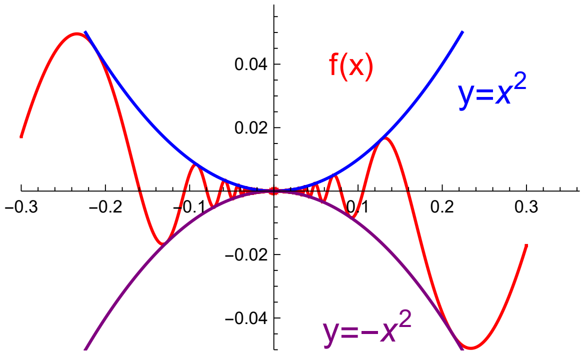
Statement. If f is differentiable on $[a, a + \delta)$ $(\delta > 0)$ and f' has a discontinuity at a then the limit
$\lim_{x \to a+0} f(x)$ doesn't exist or $\exists \lim_{x \to a+0} f(x) \notin \mathbb{R}$.

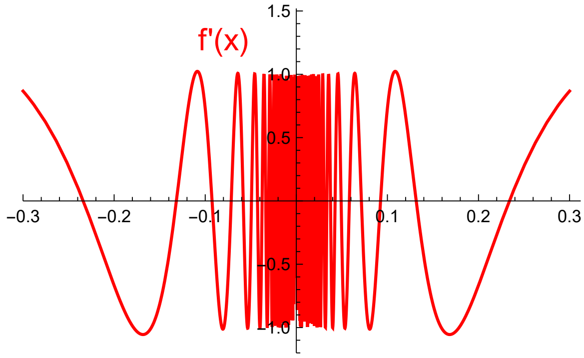
Continuously differentiable functions

Definition. Assume that I is a neighbourhood of $a \in D_f$ and f is differentiable on $I \cap D_f$.
Then f is **continuous differentiable at *a*** if f' is continuous at a.
f is **continuously differentiable** on A if f is continuous differentiable $\forall x \in A$.
Notation: $C^1(A) = \{f : f \text{ is continuously differentiable on } A\}$.

Example: The function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable but f' is not continuous

at $x_0 = 0$, since $f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.





Higher order derivatives

Definition. If f' is differentiable at x then we say that f is twice differentiable at x and the second derivative or second order derivative of f at x_0 is $f''(x) = (f')'(x)$. Differentiating f repeatedly, we get the third, ..., nth derivative of f.

Notation: $f''(x) = f^{(2)}(x) = \frac{d^2 f(x)}{d x^2}$

$f'''(x) = f^{(3)}(x) = \frac{d^3 f(x)}{d x^3}$

...

$f^{(n)}(x) = \frac{d^n f(x)}{d x^n}$

By definition: $f^{(0)}(x) = f(x)$

Example: $f(x) = \sin x \implies f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,\ f^{(4)}(x) = \sin x,\ \ldots$

$f(x) = e^x \implies f^{(n)}(x) = e^x \quad \forall n \in \mathbb{N}$

Investigation of differentiable functions

Monotonicity on an interval

Theorem. Assume that $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable. Then

(1) f is monotonically increasing $\iff f'(x) \geq 0$ for all $x \in (a, b)$

(2) f is monotonically decreasing $\iff f'(x) \leq 0$ for all $x \in (a, b)$

(3) f is constant $\iff f'(x) = 0$ for all $x \in (a, b)$

(4) $f'(x) > 0$ for all $x \in (a, b) \implies f$ is strictly monotonically increasing

(5) $f'(x) < 0$ for all $x \in (a, b) \implies f$ is strictly monotonically decreasing

Proof. (1)

(i) If f is monotonically increasing then f is locally monotonically increasing for all $x \in (a, b)$
$\Longrightarrow f'(x) \geq 0 \;\; \forall\, x \in (a, b)$.

(ii) Assume that $f'(x) \geq 0$ for all $x \in (a, b)$. Let $a < x_1 < x_2 < b$ and apply Lagrange's mean value theorem for $[x_1, x_2]$. Then there exists $c \in (x_1, x_2) \subset (a, b)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \geq 0 \;\Longrightarrow\; f(x_2) \geq f(x_1)$$

Therefore if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$, so f is monotonically increasing on (a, b).

(2) Similar to case (1).

(3) f is constant $\Longleftrightarrow$ f is monotonically increasing and decreasing
$\Longleftrightarrow f'(x) \geq 0$ and $f'(x) \leq 0 \;\; \forall\, x \in (a, b) \Longleftrightarrow f'(x) = 0 \;\; \forall\, x \in (a, b)$

(4) and (5): similar to case (1) (ii)

Remark. Statements (4) and (5) cannot be reversed.
For example, $f(x) = x^3$ is strictly monotonically increasing on $\mathbb{R}$, however $f'(x) > 0$ does not hold for all $x \in \mathbb{R}$, since $f'(x) = 3\,x^2 \Longrightarrow f'(0) = 0$.

Remark. If the domain of f is not an interval then the above theorem is not true, as the following examples show.
1) Let $f : \mathbb{R} \setminus \mathbb{Z} \longrightarrow \mathbb{R}$, $f(x) = \{x\} = x - [x]$. Then f is differentiable on $\mathbb{R} \setminus \mathbb{Z}$ and $f'(x) = 1 > 0$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$ but f is not monotonically increasing.
2) Let $f : \mathbb{R} \setminus \mathbb{Z} \longrightarrow \mathbb{R}$, $f(x) = [x]$. Then f is differentiable on $\mathbb{R} \setminus \mathbb{Z}$ and $f'(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$ but f is not constant.

Local extremum, sufficient conditions

Definition. If f is differentiable at x_0 and $f'(x_0) = 0$ then x_0 is a **stationary point** of f.
If $f'(x_0) = 0$ or f is not differentiable at x_0 then x_0 is a **critical point** of f.

Remark. Recall that if f is differentiable at $x_0 \in \operatorname{int} D_f$ and f has a local extremum at x_0 then $f'(x_0) = 0$.
This is a necessary condition for the existence of a local extremum.
The next two theorems formulate sufficient conditions.

Theorem (Sufficient condition for a local extremum, first derivative test).
Assume that f is differentiable at $x_0 \in \operatorname{int} D_f$.
If $f'(x_0) = 0$ and f' changes sign at x_0, then f has a local extremum at x_0.

Namely, if $f'(x_0) = 0$ and f' is (strictly) locally $\left\{ \begin{array}{l} \text{increasing} \\ \text{decreasing} \end{array} \right.$ at x_0

then f has a (strict) local $\left\{ \begin{array}{l} \text{minimum} \\ \text{maximum} \end{array} \right.$ at x_0.

Proof. Assume that $f'(x_0) = 0$ and f' is locally increasing at x_0
(that is, f' changes sign from negative to positive)

$$\Longrightarrow \exists\, \delta > 0 \text{ such that } \left\{ \begin{array}{l} f'(x) \leq 0 \;\text{ if }\; x_0 - \delta < x < x_0 \\ f'(x) \geq 0 \;\text{ if }\; x_0 < x < x_0 + \delta \end{array} \right.$$

$$\Longrightarrow \begin{cases} f \text{ is monotonically decreasing on } (x_0 - \delta, x_0) \\ f \text{ is monotonically increasing on } (x_0, x_0 + \delta) \end{cases}$$

$$\Longrightarrow \begin{cases} f(x) \geq f(x_0) \text{ if } x_0 - \delta < x < x_0 \\ f(x) \geq f(x_0) \text{ if } x_0 < x < x_0 + \delta \end{cases} \Longrightarrow f \text{ has a local minimum at } x_0.$$

Theorem (Sufficient condition for a local extremum, second derivative test).
Assume that f is twice differentiable at $x_0 \in \operatorname{int} D_f$.
If $f'(x_0) = 0$ and $f''(x_0) \neq 0$ then f has a local extremum at x_0.

$$\text{If } \begin{cases} f''(x_0) > 0 \\ f''(x_0) < 0 \end{cases} \text{ then } f \text{ has a strict local } \begin{cases} \text{minimum} \\ \text{maximum} \end{cases} \text{ at } x_0.$$

Proof. $f''(x_0) > 0 \Longrightarrow f'$ is locally increasing at x_0 and $f'(x_0) = 0$
$\Longrightarrow$ by the previous theorem f has a local minimum at x_0.

Remark. The sign change of f' at x_0 is only a sufficient but not a necessary condition for the existence of a local extremum at x_0.

For example, if $$f(x) = \begin{cases} x^2\left(2 + \sin\left(\frac{1}{x}\right)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

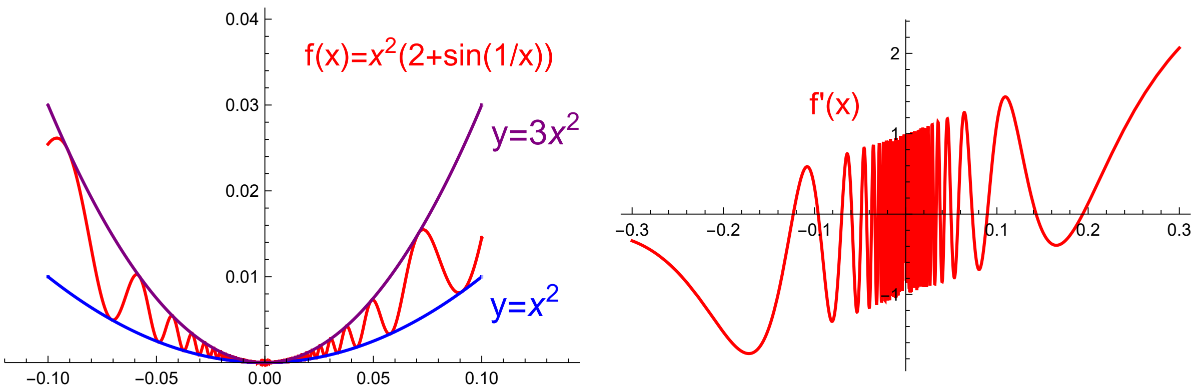
then f is differentiable for all $x \in \mathbb{R}$. At $x = 0$:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2\left(2 + \sin\left(\frac{1}{x}\right)\right)}{x} = \lim_{x\to 0} x\left(2 + \sin\left(\frac{1}{x}\right)\right) = 0 \ \text{(since it is } 0 \cdot \text{bounded)},$$

so the necessary condition holds at $x_0 = 0$.
However, in any neighbourhood of $x_0 = 0$:

f has strictly monotonic increasing and decreasing sections $\Longrightarrow$
f' has both positive and negative values $\Longrightarrow$
f' doesn’t change sign at $x_0 = 0$.

Yet f has a local extreme value at $x_0 = 0$, and it is even an absolute minimum here.



Local extremum and higher order derivatives

Remark. If $f'(x_0) = 0$ and $f''(x_0) = 0$ then it cannot be decided whether f has a local extremum at x_0. For example:

1) $f(x) = x^3$ does not have a local extremum at $x_0 = 0$,
2) $f(x) = x^4$ has a local minimum at $x_0 = 0$,
3) $f(x) = -x^4$ has a local maximum at $x_0 = 0$, and in each case $f'(0) = f''(0) = 0$.

Theorem. (1) Assume that f is $2k$ times differentiable at x_0, $k \geq 1$.

$$\text{If } f'(x_0) = .. = f^{(2k-1)}(x_0) = 0 \text{ and } \begin{cases} f^{(2k)}(x_0) > 0 \\ f^{(2k)}(x_0) < 0 \end{cases}$$

then f has a strict local $\begin{cases} \text{minimum} \\ \text{maximum} \end{cases}$ at x_0.

(2) Assume that f is $2k+1$ times differentiable at x_0, $k \geq 1$.
If $f'(x_0) = .. = f^{(2k)}(x_0) = 0$ and $f^{(2k+1)}(x_0) \neq 0$, then f is strictly monotonic in a neighbourhood of x_0, so f doesn't have a local extremum at x_0.

Proof. (1) We prove the statement for a strict local minimum by induction.

(i) If $k = 1$ then the statement is true.

(ii) Assume that the statement holds for $k-1$ and let $g = f''$.
($\Longrightarrow g' = f'''$, ..., $g^{(2k-3)} = f^{(2k-1)}$, $g^{(2k-2)} = f^{(2k)}$.)
From the induction hypothesis it follows that
if $g'(x_0) = .. = g^{(2k-3)}(x_0) = 0$ and $g^{(2k-2)}(x_0) > 0$ then the function
$g = f''$ has a strict local minimum at x_0.

(iii) We want to prove that if
$f'(x_0) = f''(x_0) = f'''(x_0) = .. = f^{(2k-1)}(x_0) = 0$ and $f^{(2k)}(x_0) > 0$ then
f has a strict local minimum at x_0.
Since $f''(x_0) = 0$ and f'' has a strict local minimum at x_0,
then $\exists\, \delta > 0$ such that $f''(x) > 0$, $\forall x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$
$\Longrightarrow f'$ is strictly monotonically increasing on $(x_0 - \delta, x_0 + \delta)$
$\Longrightarrow f'$ is strictly locally increasing at x_0
$\Longrightarrow f$ has a strict local minimum at x_0.

(2) Assume that $f'(x_0) = f''(x_0) = .. = f^{(2k)}(x_0) = 0$ and $f^{(2k+1)}(x_0) \neq 0$.
Let $g = f'$, then $g'(x_0) = .. = g^{(2k-1)}(x_0) = 0$ and $g^{(2k)}(x_0) \neq 0$.
$\Longrightarrow$ by part (1), $g = f'$ has a strict local extremum at x_0.
Since $f'(x_0) = 0$, then either $f'(x) > 0$ or $f'(x) < 0$, $\forall x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$
$\Longrightarrow$ f is strictly monotonic on $(x_0 - \delta, x_0 + \delta)$
$\Longrightarrow$ f doesn't have a local extremum at x_0

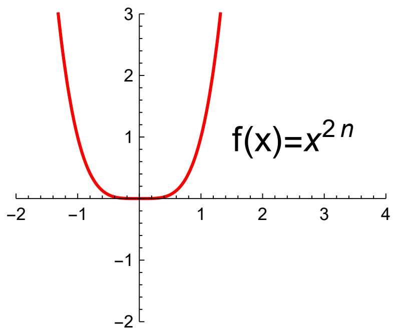
Example. $f(x) = x^n$ is n times differentiable,

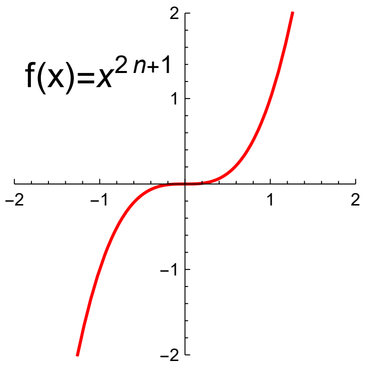
$f^{(k)}(x) = n(n-1)(n-2)\ldots(n-k+1)\,x^{n-k}$, $\quad k = 1, 2, \ldots, n-1$

$f^{(n)}(x) = n!$

$\Longrightarrow$ if $x_0 = 0$, then $f'(0) = f''(0) = .. = f^{(n-1)}(0) = 0$, $f^{(n)}(0) = n! > 0$

$\Longrightarrow$ at $x_0 = 0$ f has a local minimum if n is even and f doesn't have a local extremum if n is odd.





Convexity / concavity on an interval

Theorem (Necessary and sufficient condition for convexity).

If f is differentiable on the interval I, then the following statements are equivalent.

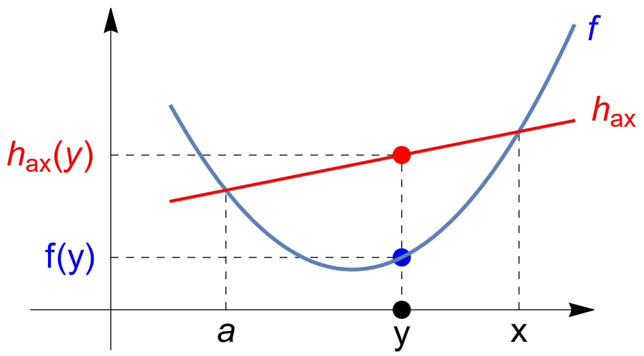
(1) f is convex on I

(2) $f(x) \geq f(a) + f'(a)(x-a)$ if $x, a \in I$

(3) f' is monotonically increasing on I

Remark. The geometrical meaning of (2) is that for all $a \in I$, the graph of f lies above the tangent line at a.

Proof of (1) $\Longrightarrow$ (2):



$y = \lambda a + (1-\lambda) x$

$h_{ax}(y) = \lambda f(a) + (1-\lambda) f(x)$

f is convex $\Longrightarrow f(y) \leq h_{ax}(y)$

If $a < x$ and $y \in (a, x)$ then $\exists\, \lambda \in (0, 1)$ such that

$$y = \lambda a + (1-\lambda) x \Longrightarrow y - a = (\lambda - 1) a + (1-\lambda) x$$
$$\Longrightarrow \boldsymbol{y - a = (1-\lambda)(x-a)}$$

f is convex $\Longrightarrow f(y) \leq \lambda f(a) + (1-\lambda) f(x)$

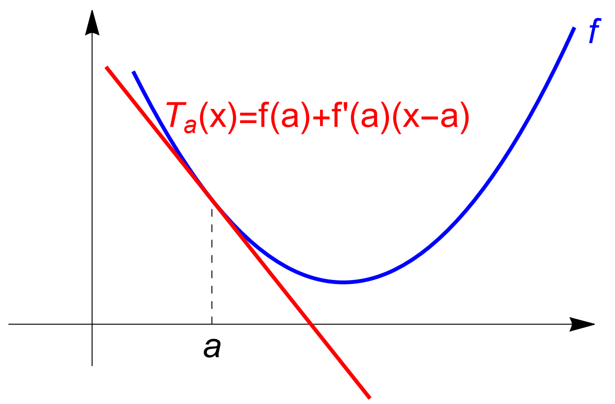
$$\Longrightarrow f(y) - f(a) \leq (\lambda - 1) f(a) + (1-\lambda) f(x)$$
$$\Longrightarrow \boldsymbol{f(y) - f(a) \leq (1-\lambda)(f(x) - f(a))}$$

Dividing both sides by $\boldsymbol{y - a = (1-\lambda)(x-a) > 0}$:

$$\frac{\boldsymbol{f(y) - f(a)}}{\boldsymbol{y-a}} \leq \frac{\boldsymbol{f(x) - f(a)}}{\boldsymbol{x-a}}$$

If $y \longrightarrow a+$, then $f'(a) \leq \dfrac{f(x) - f(a)}{x-a} \Longrightarrow f(x) \geq f(a) + f'(a)(x-a)$ if $x, a \in I$.

If $a > x$ then the proof is similar and if $a = x$ then the statement is obvious.



Proof of (2) ⟹ (3):

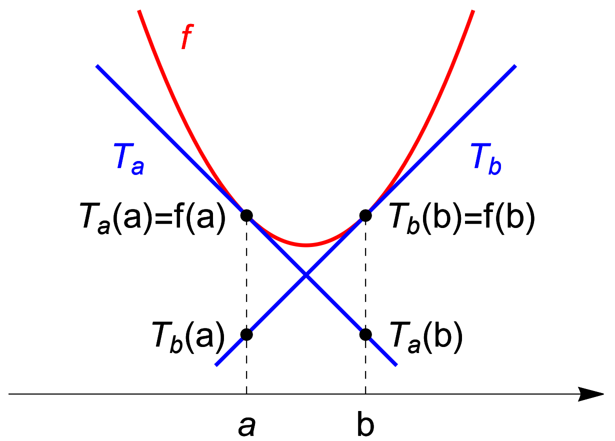
Let $T_a(x) = f(a) + f'(a)(x - a)$.

If $a, b \in I$, $a < b \implies T_a(a) = f(a) \geq T_b(a)$ and

$T_a(b) \leq f(b) = T_b(b)$

$$\implies f'(a) = \frac{T_a(b) - T_a(a)}{b - a} = \frac{T_a(b) - f(a)}{b - a} \leq \frac{f(b) - T_b(a)}{b - a} = \frac{T_b(b) - T_b(a)}{b - a} = f'(b)$$

$\implies f'$ is monotonically increasing on I



Proof of (3) ⟹ (1):

Let $a, b \in I$, $a < b$, $\lambda \in (0, 1)$ for which $x = \lambda a + (1 - \lambda) b$

$\implies$ $\boldsymbol{x - a = (1 - \lambda)(b - a)}$

$\boldsymbol{b - x = \lambda(b - a)}$

Then by Lagrange's mean value theorem there exist

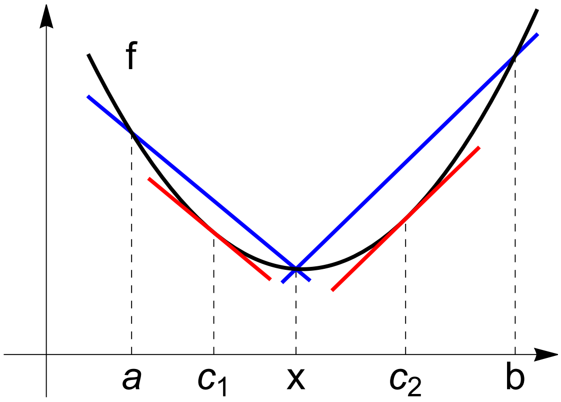
$c_1 \in (a, x)$ and $c_2 \in (x, b)$ such that $\dfrac{f(x) - f(a)}{x - a} = f'(c_1)$ and $f'(c_2) = \dfrac{f(b) - f(x)}{b - x}$.

f' is monotonically increasing $\implies f'(c_1) \leq f'(c_2)$

$$\implies \frac{f(x) - f(a)}{\boldsymbol{x - a}} \leq \frac{f(b) - f(x)}{\boldsymbol{b - x}}$$

$$\implies \frac{f(x) - f(a)}{\boldsymbol{(1 - \lambda)(b - a)}} \leq \frac{f(b) - f(x)}{\boldsymbol{\lambda(b - a)}}$$

$\implies f(x) \leq \lambda f(a) + (1 - \lambda) f(b) \implies f$ is convex on I.



Consequence (Necessary and sufficient condition for convexity).

Assume that f is twice differentiable on the interval I. Then

(1) $f''(x) \geq 0 \ \forall x \in I$ if and only if f is convex on I.

(2) $f''(x) \leq 0 \ \forall x \in I$ if and only if f is concave on I.

Consequence.

Assume that f is twice differentiable on the interval I. Then

(1) If $f''(x) > 0 \ \forall x \in I$ then f is strictly convex on I.

(2) If $f''(x) < 0 \ \forall x \in I$ then f is strictly concave on I.

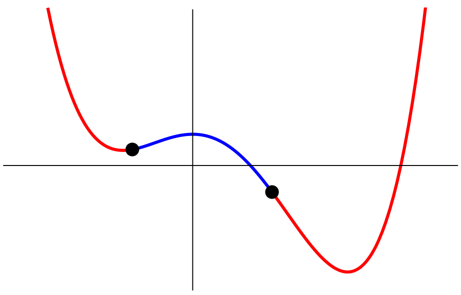
Inflection point

Definition. Assume that f is continuous at $a \in \operatorname{int} D_f$ and there exists $\delta > 0$ such that

f is convex on $(a - \delta, a)$ and concave on $(a, a + \delta)$

or f is concave on $(a - \delta, a)$ and convex on $(a, a + \delta)$.

Then a is called a point of inflection of the function f.



Theorem (Necessary condition for an inflection point, second derivative test).

If f is twice differentiable at x_0 and f has an inflection point at x_0 then $f''(x_0) = 0$.

Proof. If f is convex on $(x_0 - \delta, x_0]$ and concave on $[x_0, x_0 + \delta)$ then

f' is monotonically increasing on $(x_0 - \delta, x_0]$ and monotonically decreasing on $[x_0, x_0 + \delta)$

$\Longrightarrow$ f' has a local maximum at x_0 $\Longrightarrow$ $f''(x_0) = 0$.

Theorem (Sufficient condition for an inflection point, second derivative test).

If f is twice differentiable in a neighbourhood of x_0,

$f''(x_0) = 0$ and f'' changes sign at x_0,

then f has an inflection point at x_0.

Theorem (Sufficient condition for an inflection point, third derivative test).
If f is three times differentiable in a neighbourhood of x_0,
$f''(x_0) = 0$ and $f'''(x_0) \neq 0$,
then f has an inflection point at x_0.

Inflection point and higher order derivatives

Theorem. (1) Assume that f is $2k+1$ times differentiable at x_0, $k \geq 1$.
If $f''(x_0) = .. = f^{(2k)}(x_0) = 0$ and $f^{(2k+1)}(x_0) \neq 0$
then f has an inflection point at x_0.
(2) Assume that f is $2k$ times differentiable at x_0, $k \geq 1$.
If $f''(x_0) = .. = f^{(2k-1)}(x_0) = 0$ and $f^{(2k)}(x_0) \neq 0$, then f is strictly convex or concave in a neighbourhood of x_0, so f doesn't have an inflection point at x_0.

Linear asymptotes

Definition. The straight line $x = a$ is a **vertical asymptote** of the function f if

$$\lim_{x \to a+} f(x) = \pm\infty \text{ or } \lim_{x \to a-} f(x) = \pm\infty.$$

Definition. The straight line $g(x) = A\,x + B$ is a **linear asymptote** of the function f at ∞ or $-\infty$ if

$$\lim_{x \to \infty} (f(x) - g(x)) = 0 \text{ or } \lim_{x \to -\infty} (f(x) - g(x)) = 0.$$

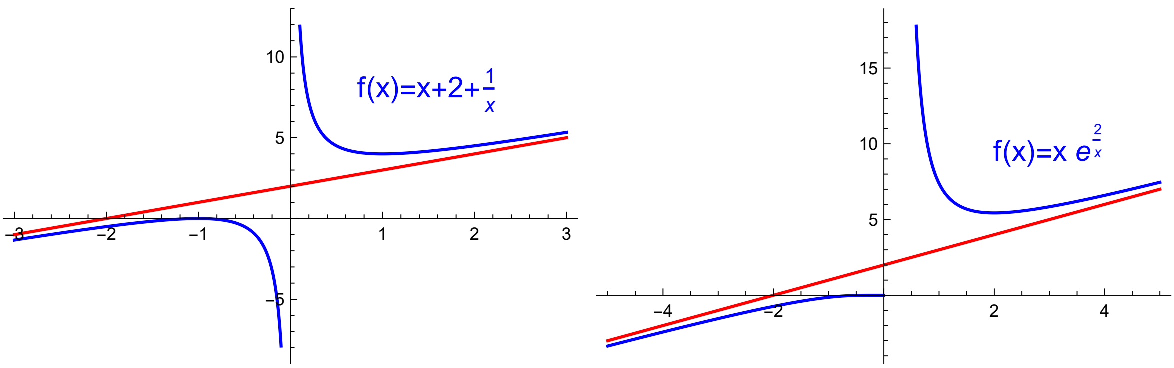
$g(x)$ is a **horizontal asymptote** if $A = 0$ and an **oblique or slant asymptote** if $A \neq 0$.

Statement. $g(x) = A\,x + B$ is a linear asymptote of f at $\pm\infty$ if and only if

$$A = \lim_{x \to \pm\infty} \frac{f(x)}{x} \quad \text{and} \quad B = \lim_{x \to \pm\infty} (f(x) - A\,x)$$

Example. $\lim_{x \to \frac{\pi}{2}\pm} \tan x = \mp\infty \implies x = \frac{\pi}{2}$ is a vertical asymptote of $f(x) = \tan(x)$.

Example. If $f(x) = x + 2 + \frac{1}{x}$ then $g(x) = x + 2$ is a linear asymptote of f at $\pm\infty$.



Example. If $f(x) = x\,e^{\frac{2}{x}}$ then $g(x) = x + 2$ is a linear asymptote of f at $\pm\infty$.

Solution. $A = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x\,e^{\frac{2}{x}}}{x} = \lim_{x \to \pm\infty} e^{\frac{2}{x}} = e^0 = 1$

$B = \lim\limits_{x\to\pm\infty}\left(x\, e^{\frac{2}{x}} - x\right) = \lim\limits_{x\to\pm\infty} \dfrac{e^{\frac{2}{x}} - 1}{\frac{1}{x}}$. Let $y = \dfrac{2}{x}$, then $B = \lim\limits_{y\to 0\pm} \dfrac{e^{y} - 1}{\frac{1}{2}\cdot y} = 2$,

using that $\lim\limits_{x\to 0} \dfrac{e^{x} - 1}{x} = 1$. The limit can also be calculate with the L'Hospital's rule.

So $g(x) = x + 2$.

Extreme values on a closed interval

Remark. If f is continuous on a closed and bounded interval then by the Weierstrass extreme value theorem f has a minimum and a maximum.

The possible points are:

1) the points where f is not differentiable

2) the points where the derivative of f is 0

3) the endpoints of the interval

Finally the largest and smallest of the possible values must be selected.