17th and 18th lectures

The exponential function

Definition. The function $f(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ is called the exponential function of base e.

Notation: e^x, $\exp_e(x)$ or $\exp(x)$.

Statement. $e^{x+y} = e^x e^y \quad \forall\, x, y \in \mathbb{R}$.

Proof. Using the identity $a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$ and choosing n large enough such that

$1+\frac{x+y}{n} > 0$, $1+\frac{x}{n} > 0$ and $1+\frac{y}{n} > 0$, we get that

$$\left|\left(1+\frac{x+y}{n}\right)^n - \left(1+\frac{x}{n}\right)^n\left(1+\frac{y}{n}\right)^n\right| = \frac{|x\,y|}{n^2}\sum_{k=0}^{n-1}\left(1+\frac{x+y}{n}\right)^k\left(\left(1+\frac{x}{n}\right)\cdot\left(1+\frac{y}{n}\right)\right)^{n-1-k}.$$

Here

$$\left(1+\frac{a}{n}\right)^k \le \begin{cases} 1 & \text{if } a \le 0 \\ e^a & \text{if } a > 0 \end{cases}$$

So

$$\left(1+\frac{x+y}{n}\right)^k\left(\left(1+\frac{x}{n}\right)\cdot\left(1+\frac{y}{n}\right)\right)^{n-1-k} \le K$$

where $K = \max\{1, e^{x+y}\}\cdot\max\{1, e^x\}\cdot\max\{1, e^y\}$, therefore

$$\left|\left(1+\frac{x+y}{n}\right)^n - \left(1+\frac{x}{n}\right)^n\left(1+\frac{y}{n}\right)^n\right| \le \frac{|x\,y|}{n^2}\cdot n\,K = \frac{K\,|x\,y|}{n} \xrightarrow{n\to\infty} 0.$$

Statement. If $x \in \mathbb{R}$, then $e^x > 0$, $e^x \ge 1+x$, and if $x < 1$, then $e^x \le \frac{1}{1-x}$.

Proof. 1) If $x \ge 0$ then from the definition it follows that $e^x > 0$.

If $x < 0$ then $e^x = \frac{1}{e^{-x}} > 0$, since $e^{-x} > 0$.

2) If $n \in \mathbb{N}^+$ such that $n \ge -x$, then $\frac{x}{n} \ge -1$, so by the Bernoulli inequality

$$\left(1+\frac{x}{n}\right)^n \ge 1 + n\cdot\frac{x}{n} = 1+x$$

By the monotonicity of the limit $e^x \ge 1+x$.

3) If $x < 1$ then $e^{-x} \ge 1+(-x) > 0 \implies e^x = \frac{1}{e^{-x}} \le \frac{1}{1-x}$.

Statement. $f(x) = e^x$ is continuous at 0.

Proof. If $x < 1$ then $1 + x \le e^x \le \frac{1}{1-x}$, so from the sandwich theorem $\lim_{x\to 0} e^x = e^0 = 1$.

Consequence. $f(x) = e^x$ is continuous.

Proof. $\lim\limits_{x\to x_0} e^x = e^{x_0} \lim\limits_{x\to x_0} e^{x-x_0} = e^{x_0} \lim\limits_{x\to 0} e^x = e^{x_0}$.

Statement. $f(x) = e^x$ is strictly monotonically increasing and its range is $(0, \infty)$.

Proof. 1) Let $x,\ y \in \mathbb{R}$ such that $x < y$. We have to show that $e^x < e^y$.

Since $y - x > 0$ then $e^{y-x} \geq 1 + (y - x) > 1$

and since $e^x > 0$ then $e^y = e^{(y-x)+x} = e^{y-x}\, e^x > 1 \cdot e^x = e^x$.

2) $\sup R_f = \infty$. Since $e^x \geq 1 + x$ and $\lim\limits_{x\to 0}(1 + x) = \infty$, so $\lim\limits_{x\to 0} e^x = \infty$.

3) $\inf R_f = 0$. Since $f(x) = e^x$ is strictly monotonically increasing, then

$$\lim_{x\to -\infty} e^x = \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

4) By the intermediate value theorem the range of f is an interval, so $R_f = (0, \infty)$.

Definition. Denote $\ln = \log_e$ the inverse of $f(x) = e^x$, so $e^{\ln x} = \ln e^x = x$.

$D_{\ln} = R_{\exp} = (0, \infty)$ and $R_{\ln} = D_{\exp} = \mathbb{R}$.

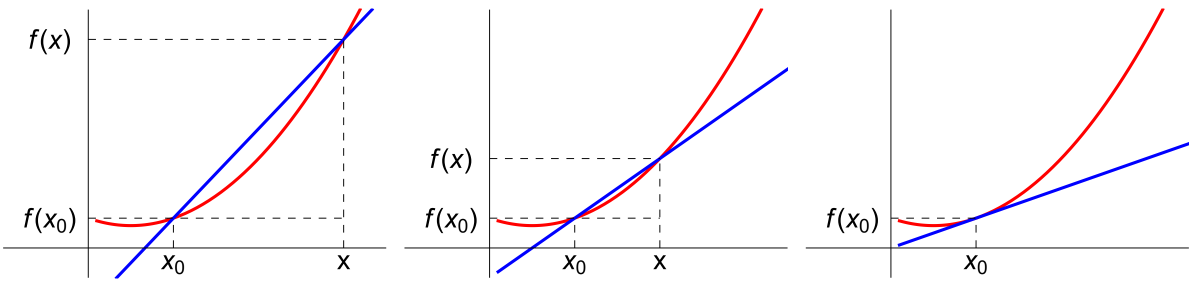
Differentiation

The derivative

Definition. Suppose that x_0 is an interior point of D_f. Then the function f is **differentiable** at x_0 if the following finite limit exists:

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The number $f'(x_0) = \frac{\mathrm{df}}{\mathrm{dx}}(x_0) \in \mathbb{R}$ is called the derivative of f at x_0.



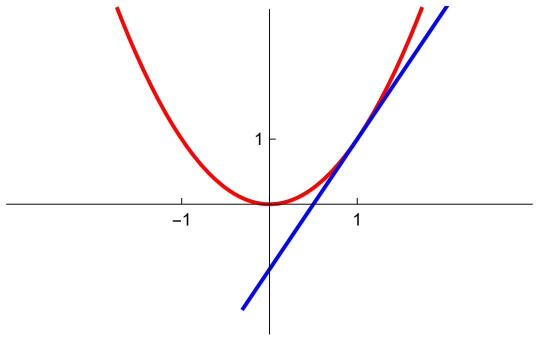
Remark. $f'(x_0)$ gives the slope of the tangent line of the graph of f at the point $(x_0, f(x_0))$.
The equation of the tangent line is $y = f(x_0) + f'(x_0)(x - x_0)$

Examples. 1) $f(x) = c \implies f'(x_0) = \lim\limits_{x\to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \lim\limits_{x\to x_0} \dfrac{c - c}{x - x_0} = 0 \quad \forall\, x_0 \in \mathbb{R}$.

2) $f(x)=x \quad \Longrightarrow \quad f'(x_0)=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}=\lim\limits_{x\to x_0}\dfrac{x-x_0}{x-x_0}=1 \qquad \forall\, x_0\in\mathbb{R}.$

3) $f(x)=x^2 \quad \Longrightarrow \quad f'(x_0)=\lim\limits_{x\to x_0}\dfrac{x^2-x_0^2}{x-x_0}=\lim\limits_{x\to x_0}(x+x_0)=2\,x_0 \qquad \forall\, x_0\in\mathbb{R}.$

Find the tangent line of f at $x_0=1$. Then $f(1)=1$, $f'(1)=2$,
so the tangent line is $y=f(1)+f'(1)(x-1)=1+2(x-1)=2\,x-1$.



One-sided derivatives

Definition. The left-hand and right-hand derivative f at a are

$$f_-'(a)=\lim_{x\to a-0}\frac{f(x)-f(a)}{x-a} \quad \text{and} \quad f_+'(a)=\lim_{x\to a+0}\frac{f(x)-f(a)}{x-a}$$

respectively, if these finite limits exist.

Theorem. Assume that $a\in \operatorname{int} D_f$. Then f is differentiable at a if and only if
$f'(a)=f_-'(a)=f_+'(a)$

Definition. Let $a<b$. Then f is differentiable on (a,b) if f is differentiable at x for all $x\in(a,b)$.
f is differentiable on $[a,b]$ if f is differentiable on (a,b) and $\exists\, f_+'(a),\ f_-'(a)\in\mathbb{R}$.
The derivative function of f is the function $f':\{x\in D_f:\exists\, f'(x)\},\ x\mapsto f'(x)$

Relation to continuity

Theorem. If f is differentiable at x_0 then f is continuous at x_0.

Proof. $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0}\left(\dfrac{f(x)-f(x_0)}{x-x_0}(x-x_0)+f(x_0)\right)=f'(x_0)\cdot 0+f(x_0)=f(x_0).$

Remark. Continuity is necessary for differentiability but not sufficient.

For example, let $f(x)=|\,x\,|=\begin{cases} x & \text{if } x\ge 0\\ -x & \text{if } x<0\end{cases}$. Then at $x_0=0$: $\dfrac{f(x)-f(0)}{x-0}=\begin{cases} 1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$

$\Longrightarrow\ f_+'(x_0)=\lim\limits_{x\to x_0+0}\dfrac{f(x)-f(x_0)}{x-x_0}=1$ and $f_-'(x_0)=\lim\limits_{x\to x_0-0}\dfrac{f(x)-f(x_0)}{x-x_0}=-1$

$\Longrightarrow\ f$ is not differentiable at $x_0=0$.

Some interesting examples.

1) The following function is everywhere continuous but nowhere differentiable:

$$f(x)=\lim_{n\to\infty}\left(\frac{1}{2}\sin 2\,x+\frac{1}{4}\sin 4\,x+\ldots+\frac{1}{2^n}\sin(2^n\,x)\right)$$

2) The following function is differentiable only at $x_0 = 0$ but discontinuous for all $x \in \mathbb{R} \setminus \{0\}$:

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ -x^2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then $\left| \dfrac{f(x) - f(0)}{x - 0} \right| = \left| \dfrac{x^2}{x} \right| = |x| \longrightarrow 0$ if $x \longrightarrow 0$

$\Longrightarrow$ f is differentiable at $x_0 = 0$ and $f'(0) = 0$ but f is discontinuous if $x \neq 0$.

Examples

Statement. $f(x) = x^n$ $(n \in \mathbb{N}^+)$ is differentiable on $\mathbb{R}$ and $f'(x) = n\,x^{n-1}$.

Proof. $f'(a) = \lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = \lim\limits_{x \to a} \left(x^{n-1} + x^{n-2}\,a + \ldots + x\,a^{n-2} + a^{n-1}\right) = n\,a^{n-1}$

Statement. $f(x) = \sin x$ is differentiable on $\mathbb{R}$ and $f'(x) = \cos x$.

Proof. $f'(a) = \lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x - a} = \lim\limits_{x \to a} \dfrac{2 \sin \frac{x-a}{2} \cdot \cos \frac{x+a}{2}}{x - a} = \lim\limits_{x \to a} \dfrac{\sin \frac{x-a}{2}}{\frac{x-a}{2}} \cdot \cos \dfrac{x + a}{2} = 1 \cdot \cos a = \cos a$

Statement. $f(x) = \cos x$ is differentiable on $\mathbb{R}$ and $f'(x) = -\sin x$.

Proof. $f'(a) = \lim\limits_{x \to a} \dfrac{\cos x - \cos a}{x - a} = \lim\limits_{x \to a} \dfrac{-2 \sin \frac{x-a}{2} \cdot \sin \frac{x+a}{2}}{x - a} = \lim\limits_{x \to a} \dfrac{-\sin \frac{x-a}{2}}{\frac{x-a}{2}} \cdot \sin \dfrac{x + a}{2} = -1 \cdot \sin a = -\sin a$

Statement. $f(x) = e^x$ is differentiable on $\mathbb{R}$ and $f'(x) = e^x$.

Proof. If $x < 1$ then $1 + x \leq e^x \leq \dfrac{1}{1 - x}$ $\Longrightarrow$ $1 \leq \dfrac{e^x - 1}{x} \leq \left(\dfrac{1}{1 - x} - 1\right) \cdot \dfrac{1}{x} = \dfrac{1}{1 - x}$

$\Longrightarrow 1 \leq \lim\limits_{x \to 0} \dfrac{e^x - 1}{x} \leq \lim\limits_{x \to 0} \dfrac{1}{1 - x} = 1 \Longrightarrow \boldsymbol{\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1}$

$\Longrightarrow f'(a) = \lim\limits_{x \to a} \dfrac{e^x - e^a}{x - a} = e^a \lim\limits_{x \to a} \dfrac{e^{x-a} - 1}{x - a} = e^a \cdot 1 = e^a$.

Operations with the derivatives

Theorem. If f and g are differentiable at a and $c \in \mathbb{R}$ then

$(c \cdot f)$, $(f \pm g)$ and $(f \cdot g)$ are differentiable at a and

(1) $(c\,f)'(a) = c \cdot f'(a)$

(2) $(f \pm g)'(a) = f'(a) \pm g'(a)$

(3) $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$

If $g(a) \neq 0$ then $\dfrac{1}{g}$ and $\dfrac{f}{g}$ are differentiable at a and

(4) $\left(\dfrac{1}{g}\right)'(a) = -\dfrac{g'(a)}{g^2(a)}$

(5) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{g^2(a)}$

Proof. (1) $(c\,f)'(a) = \lim\limits_{x\to a}\frac{(c\cdot f)(x) - (c\cdot f)(a)}{x-a} = \lim\limits_{x\to a}\frac{c\cdot f(x) - c\cdot f(a)}{x-a} = \lim\limits_{x\to a} c\cdot\frac{f(x)-f(a)}{x-a} = c\cdot f'(a)$

(2) $(f+g)'(a) = \lim\limits_{x\to a}\frac{(f+g)(x) - (f+g)(a)}{x-a} = \lim\limits_{x\to a}\left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right) = f'(a) + g'(a)$

(3) $(f\cdot g)'(a) = \lim\limits_{x\to a}\frac{(f\cdot g)(x) - (f\cdot g)(a)}{x-a} = \lim\limits_{x\to a}\frac{f(x)\cdot g(x) - f(a)\cdot g(x) + f(a)\cdot g(x) - f(a)\cdot g(a)}{x-a} =$

$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot g(x) + f(a)\cdot\frac{g(x)-g(a)}{x-a}\right) = f'(a)\cdot g(a) + f(a)\cdot g'(a)$$

(4) $\left(\frac{1}{g}\right)'(a) = \lim\limits_{x\to a}\frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} = \lim\limits_{x\to a}\frac{g(a)-g(x)}{g(x)\,g(a)\,(x-a)} = \lim\limits_{x\to a}\frac{\frac{g(a)-g(x)}{x-a}}{g(x)\,g(a)} = -\frac{g'(a)}{g^2(a)}$

(5) $\left(\frac{f}{g}\right)'(a) = \left(f\cdot\frac{1}{g}\right)'(a) = f'(a)\cdot\left(\frac{1}{g}\right)(a) + f(a)\cdot\left(-\frac{g'(a)}{g^2(a)}\right) = \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{g^2(a)}$

Examples

Statement. $(\tan x)' = \frac{1}{\cos^2 x}$ and $(\cot x)' = -\frac{1}{\sin^2 x}$

Proof. $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}$

$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin x\cdot\sin x - \cos x\cdot\cos x}{\sin^2 x} = -\frac{1}{\sin^2 x}$

Linear approximation

Theorem. The function f is differentiable at a if and only if it can be approximated by a linear function at a, that is, there exists $A\in\mathbb{R}$ (independent of x) such that

$$f(x) = f(a) + A(x-a) + \varepsilon(x)\,(x-a),\quad \text{where } \lim_{x\to a}\varepsilon(x) = 0.$$

Then $A = f'(a)$.

Proof. 1) Assume that f is differentiable at a and let $\varepsilon(x) = \frac{f(x)-f(a)}{x-a} - f'(a)$.

$\Longrightarrow f(x) = f(a) + f'(a)\,(x-a) + \varepsilon(x)\,(x-a)$ and $\lim\limits_{x\to a}\varepsilon(x) = 0$.

2) Assume that $f(x) = f(a) + A(x-a) + \varepsilon(x)\,(x-a)$ and $\lim\limits_{x\to a}\varepsilon(x) = 0$.

$$\Longrightarrow \frac{f(x)-f(a)}{x-a} = A + \varepsilon(x) \longrightarrow A \text{ if } x \longrightarrow a$$

$\Longrightarrow$ f is differentiable at a and $f'(a) = A$.

Remark. If f is differentiable at a, then $L(x) = f(a) + f'(a)(x-a)$ is the **linearization** of f at a.
The approximation $f(x) \approx L(x)$ is the **standard linear approximation** of f at a.
Then $\lim_{x \to a}(f(x) - L(x)) = 0$.

Chain rule

Theorem (Chain rule). If g is differentiable at a and f is differentiable at $g(a)$ then
$f \circ g$ is differentiable at a and $(f \circ g)' = f'(g(a)) \cdot g'(a)$.

Proof. 1) Since g is differentiable at a then there exists $\varepsilon_1 : D_g \longrightarrow \mathbb{R}$ such that
$\mathbf{g(x) - g(a) = g'(a)(x-a) + \varepsilon_1(x)(x-a)}$ and $\lim_{x \to a} \varepsilon_1(x) = 0$.

2) Since f is differentiable at $g(a)$ then there exists $\varepsilon_2 : D_f \longrightarrow \mathbb{R}$ such that
$f(\mathbf{t}) - f(g(a)) = f'(g(a))(\mathbf{t} - g(a)) + \varepsilon_2(\mathbf{t})(\mathbf{t} - g(a))$ and $\lim_{\mathbf{t} \to g(a)} \varepsilon_2(\mathbf{t}) = 0$.

3) Substituting $\mathbf{t = g(x)}$:
$f(g(x)) - f(g(a)) = f'(g(a))\,\mathbf{(g(x) - g(a))} + \varepsilon_2(g(x))\,\mathbf{(g(x) - g(a))} =$
$= f'(g(a))\,\mathbf{(g'(a)(x-a) + \varepsilon_1(x)(x-a))} + \varepsilon_2(g(x))\,\mathbf{(g'(a)(x-a) + \varepsilon_1(x)(x-a))} =$
$= f'(g(a))\,g'(a)(x-a) + \varepsilon(x)(x-a)$
where
$$\varepsilon(x) = f'(g(a))\,\varepsilon_1(x) + \varepsilon_2(g(x))\,g'(a) + \varepsilon_2(g(x))\,\varepsilon_1(x)$$
If $x \to a$ then $\varepsilon(x) \longrightarrow 0$, so $f \circ g$ can be linearly approximated at a
$\Longrightarrow$ $f \circ g$ is differentiable at a and we obtain the chain rule.

Derivative of the inverse

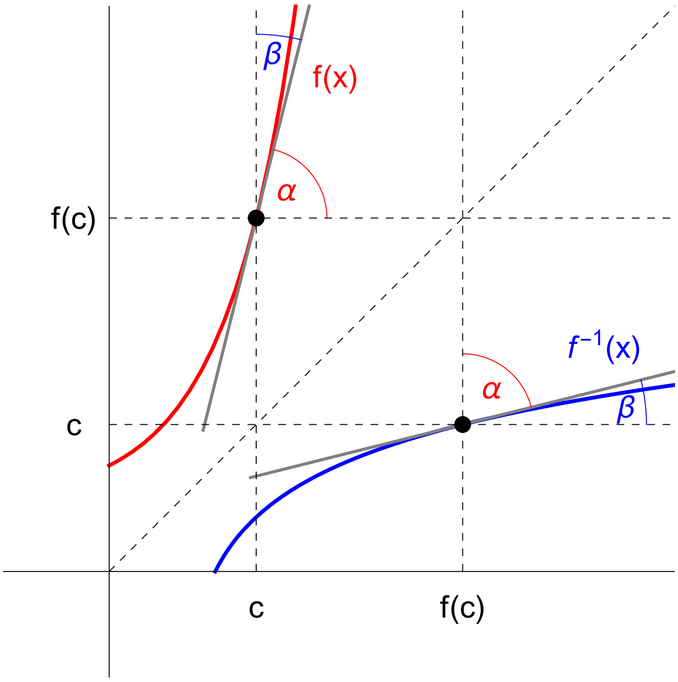
Theorem. Assume that f is continuous and strictly monotonic on (a, b),
f is differentiable at $c \in (a, b)$ and $f'(c) \neq 0$. Then f^{-1} is differentiable at $f(c)$ and
$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}$$

Proof. 1) Let $\varphi(x) = f^{-1}(x) \Longrightarrow \boldsymbol{\varphi(f(c)) = c}$ and $\mathbf{f(\boldsymbol{\varphi}(y)) = y}$ $\forall\, y \in f((a, b))$.

2) Let $F(x) = \dfrac{f(x) - f(c)}{x - c}$. Then $\dfrac{\varphi(y) - \boldsymbol{\varphi(f(c))}}{\mathbf{y} - f(c)} = \dfrac{\varphi(y) - \mathbf{c}}{\mathbf{f(\boldsymbol{\varphi}(y))} - f(c)} = \dfrac{1}{F(\varphi(y))}$

3) φ is strictly monotonic $\Longrightarrow$ if $y \neq f(c)$ then $\varphi(y) \neq c$

4) If $y \neq f(c)$ then $\varphi'(f(c)) = \lim_{y \to f(c)} \dfrac{\varphi(y) - \varphi(f(c))}{y - f(c)} = \lim_{y \to c} \dfrac{1}{F(y)} = \dfrac{1}{f'(c)}$

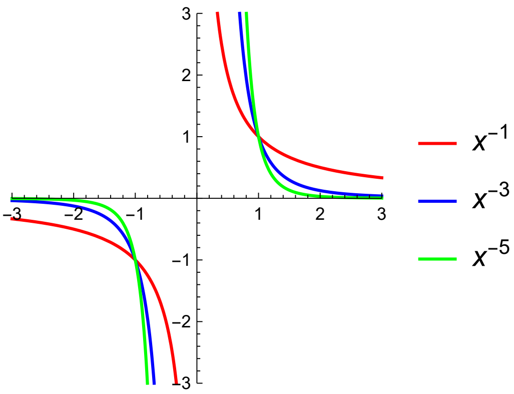


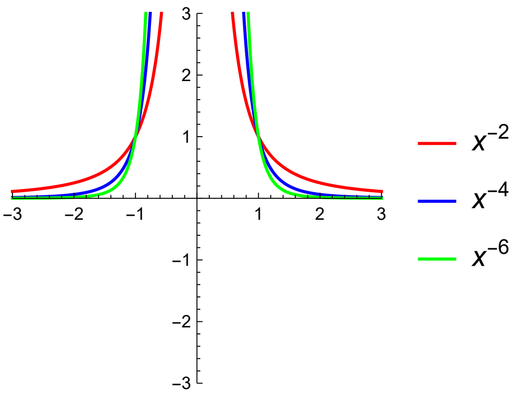
Remark. $\alpha+\beta=\frac{\pi}{2} \implies \tan\alpha\tan\beta=1 \implies \left(f^{-1}\right)'(f(c))=\tan\beta=\frac{1}{\tan\alpha}=\frac{1}{f'(c)}$

Remark. $f\left(f^{-1}(x)\right)=x \implies f'\left(f^{-1}(x)\right)\cdot\left(f^{-1}\right)'(x)=1 \implies \left(f^{-1}\right)'(x)=\frac{1}{f'\left(f^{-1}(x)\right)}$

Examples

Statement. Let $f(x)=x^{-n}$ $(n\in\mathbb{N}^+,\ x\neq 0) \implies f'(x)=-n\,x^{-n-1}$

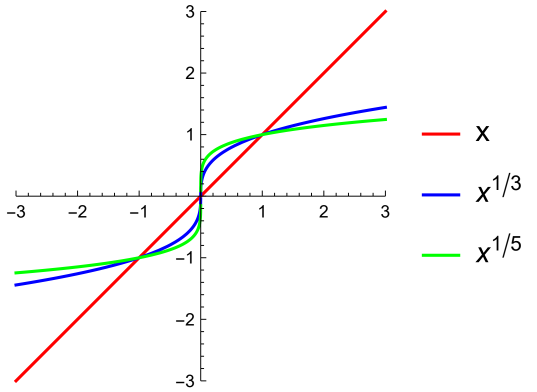


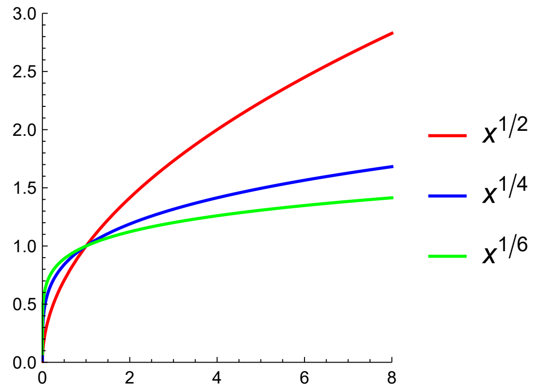


Proof: $\left(\frac{1}{g(x)}\right)'=-\frac{g'(x)}{g^2(x)} \implies f'(x)=\left(x^{-n}\right)'=\left(\frac{1}{x^n}\right)'=-\frac{n\,x^{n-1}}{x^{2n}}=-n\,x^{-n-1}$

Statement. Let $f(x)=\sqrt[n]{x}$ $(n\in\mathbb{N}^+)$. Then $D_f=[0,\infty)$ if n is even and $D_f=\mathbb{R}$ if n is odd.

$\implies f'(x)=\frac{1}{n}x^{\frac{1}{n}-1}$ where $x>0$ if n is even and $x\neq 0$ if $n>1$ is odd.





Proof. Using the derivative of the inverse:

$$f(x) = y = \sqrt[n]{x} \implies x = f^{-1}(y) = y^n, \quad (f^{-1})'(y) = n\, y^{n-1}$$

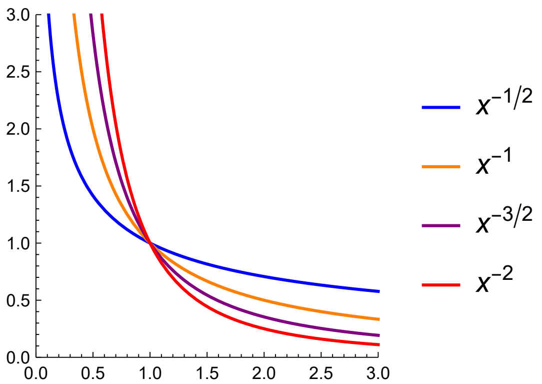
$$\implies f'(x) = \frac{1}{(f^{-1})'(y)\,|_{y=f(x)}} = \frac{1}{n \cdot y^{n-1}} = \frac{1}{n\left(\sqrt[n]{x}\right)^{n-1}} = \frac{1}{n\, x^{1-\frac{1}{n}}} = \frac{1}{n}\, x^{\frac{1}{n}-1}$$

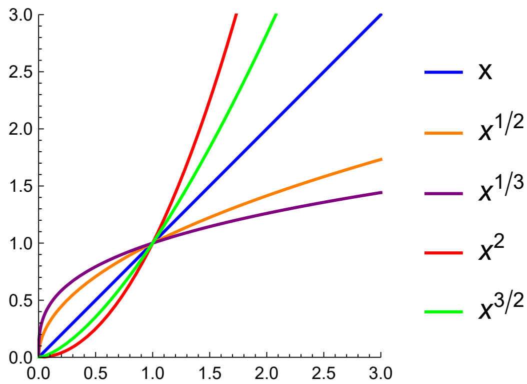
If $n > 1$ is odd then $f'(0)$ doesn't exist and if n is even then $f_+'(0)$ doesn't exist. (The tangent line at 0 is vertical.)

Statement. $f(x) = x^{\frac{p}{q}}$ $(p \in \mathbb{Z},\, q \in \mathbb{N}^+,\, x > 0)$ is differentiable and $f'(x) = \frac{p}{q}\, x^{\frac{p}{q}-1}$.

Proof. Using the chain rule: $f'(x) = \frac{1}{q}\,(x^p)^{\frac{1}{q}-1} \cdot p\, x^{p-1} = \frac{p}{q}\, x^{\frac{p}{q}-1}$

Statement. $f(x) = x^\alpha$ $(\alpha \in \mathbb{R},\, x > 0)$ is differentiable and $f'(x) = \alpha\, x^{\alpha-1}$.





Proof. Using the chain rule: $f'(x) = (x^\alpha)' = \left(e^{\alpha \ln x}\right)' = e^{\alpha \ln x} \cdot \alpha \cdot \frac{1}{x} = x^\alpha \cdot \frac{\alpha}{x} = \alpha\, x^{\alpha-1}$

Statement. $f(x) = a^x$ is differentiable for all $x \in \mathbb{R}$ and $f'(x) = a^x \cdot \ln a$.

Proof. Using the chain rule: $f'(x) = (a^x)' = \left(e^{x \ln a}\right)' = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a$

Statement. $f(x) = \ln x$ is differentiable for all $x > 0$ and $f'(x) = \frac{1}{x}$.

Proof. Using the derivative of the inverse: $f(x) = \ln x$, $f^{-1}(x) = e^x$, $(f^{-1})'(x) = e^x$

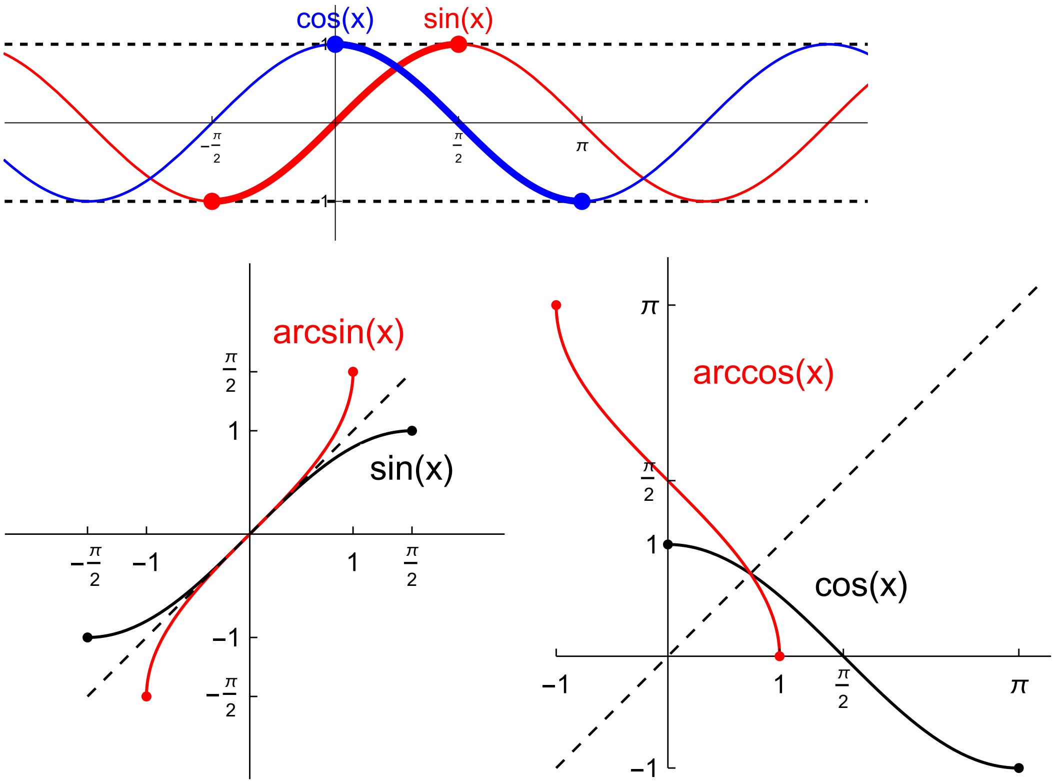
$$\implies f'(x) = \frac{1}{(f^{-1})'(f(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Statement. $f(x) = \log_a x$ $(0 < a \neq 1,\ x > 0)$ is differentiable and $f'(x) = \dfrac{1}{x \ln a}$.

Proof. $f'(x) = (\log_a x)' = \left(\dfrac{\ln x}{\ln a}\right)' = \dfrac{1}{\ln a} \cdot (\ln x)' = \dfrac{1}{\ln a} \cdot \dfrac{1}{x}$

Trigonometric functions and their inverses

Remark. The sine, cosine, tangent and cotangent functions are periodic, so they are not invertible on their whole domains. In order to define their inverses, they must be restricted to suitable intervals where they are one-to-one.



Definition. The arcsine function is the inverse of the restriction of the sine function to the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$: arcsin $= \left(\sin|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}\right)^{-1}$

The arccosine function is the inverse of the restriction of the cosine function to the interval $[0, \pi]$: arccos $= \left(\cos|_{[0,\pi]}\right)^{-1}$

$D_{\text{arcsin}} = [-1, 1]$ and $R_{\text{arcsin}} = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$D_{\text{arccos}} = [-1, 1]$ and $R_{\text{arccos}} = [0, \pi]$

The derivatives are

$$\arcsin'(x) = (\arcsin x)' = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} =$$

$$= \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}}, \quad x \in (-1, 1).$$

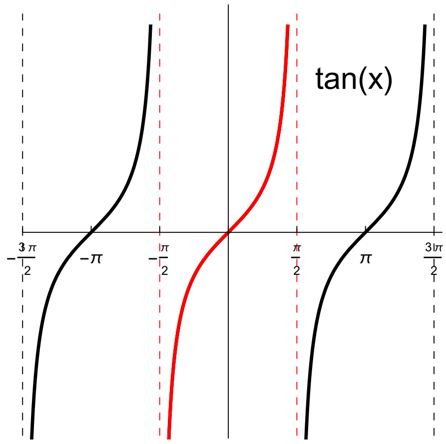
$$\arccos'(x) = (\arccos x)' = \frac{1}{\cos'(\arccos x)} = \frac{1}{-\sin(\arccos x)} =$$
$$= -\frac{1}{\sqrt{1-\cos^2(\arccos x)}} = -\frac{1}{\sqrt{1-x^2}}, \quad x \in (-1, 1).$$

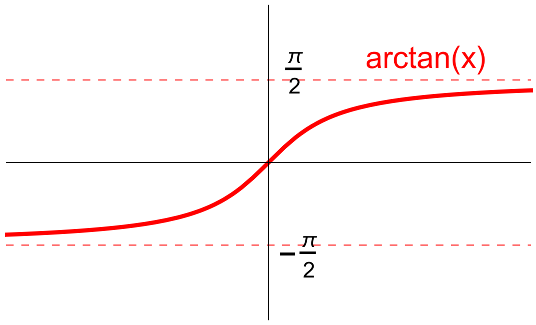
Definition. The arctangent function is the inverse of the restriction of the tangent function to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$: $\quad \arctan = \left(\tan|_{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}\right)^{-1}$

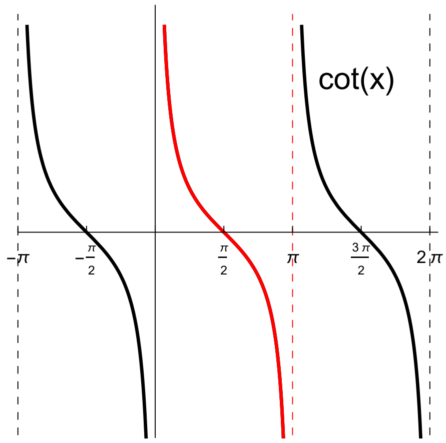
The arccotangent function is the inverse of the restriction of the cotangent function to the interval $(0, \pi)$: $\quad \text{arccot} = \left(\cot|_{(0,\pi)}\right)^{-1}$

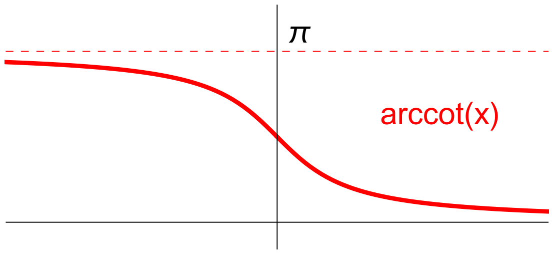
$D_{\arctan} = \mathbb{R}$ and $R_{\arctan} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$D_{\text{arccot}} = \mathbb{R}$ and $R_{\text{arccot}} = (0, \pi)$









Using that $(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x$, and $(\cot x)' = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x)$, the derivatives are

$$\arctan'(x) = (\arctan x)' = \frac{1}{\tan'(\arctan x)} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}$$

$$\text{arccot}'(x) = -\frac{1}{1+x^2}$$

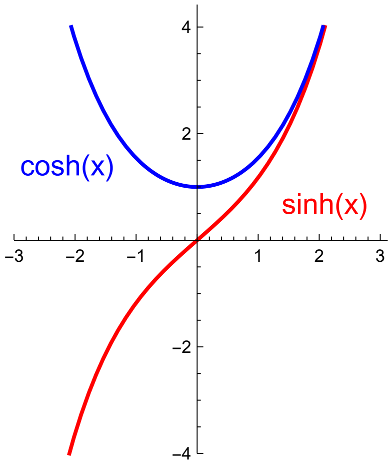
Hyperbolic functions and their inverses

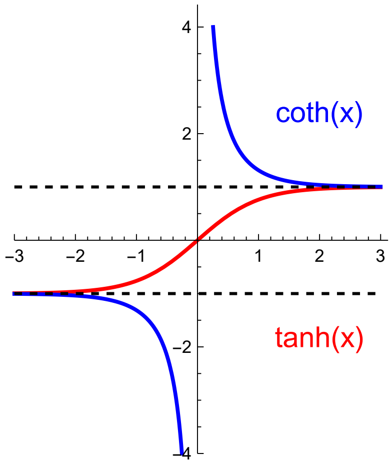
Definition: Hyperbolic sine function: $\sinh x = \dfrac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R}$

Hyperbolic cosine function: $\cosh x = \dfrac{e^x + e^{-x}}{2}, \quad x \in \mathbb{R}$

Hyperbolic tangent function: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}, \quad x \in \mathbb{R}$

Hyperbolic cotangent function: $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \in \mathbb{R} \setminus \{0\}$





Properties:

1) sinh:
- $D_{\sinh} = \mathbb{R}$, $R_{\sinh} = \mathbb{R}$
- $\lim_{x \to \pm\infty} \sinh x = \pm\infty$
- strictly monotonically increasing, continuous, odd

2) cosh:
- $D_{\cosh} = \mathbb{R}$, $R_{\cosh} = [1, \infty)$
- $\lim_{x \to \pm\infty} \cosh x = \infty$
- strictly mon. decreasing on $(-\infty, 0]$, strictly mon. increasing on $[0, \infty)$, continuous, even

3) tanh:
- $D_{\tanh} = \mathbb{R}$, $R_{\tanh} = (-1, 1)$
- $\lim_{x \to \pm\infty} \tanh x = \pm 1$
- strictly monotonically increasing, continuous, odd

4) coth :
- $D_{\coth} = \mathbb{R} \setminus \{0\}$, $R_{\coth} = (-\infty, -1) \cup (1, \infty)$
- $\lim_{x \to \pm\infty} \cosh x = \pm 1$, $\lim_{x \to 0 \pm 0} \cosh x = \pm\infty$
- strictly mon. decreasing on $(-\infty, 0)$ and $(0, \infty)$, continuous, odd

Remark. If a chain or a rope is suspended at two points, then its shape is called a catenary curve and it is the graph of the hyperbolic cosine function.
See https://en.wikipedia.org/wiki/Catenary

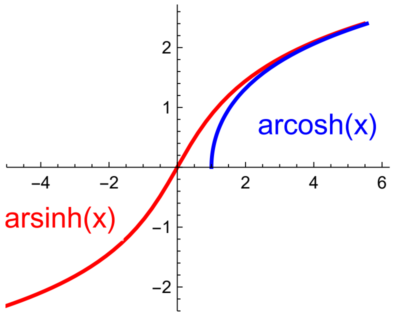
Some identities:

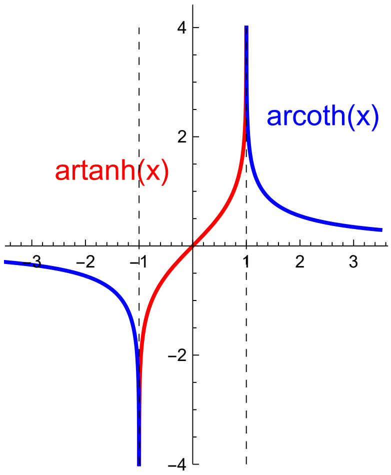
1. $\cosh^2 x - \sinh^2 x = 1$
2. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
3. $\sinh 2x = 2 \sinh x \cosh x$
4. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
5. $\cosh 2x = \cosh^2 x + \sinh^2 x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Area hyperbolic functions

The inverse hyperbolic functions are the area hyperbolic functions.

Definition: Area hyperbolic sine: $\operatorname{arsinh} = \sinh^{-1}$, $D_{\operatorname{arsinh}} = \mathbb{R}$
Area hyperbolic cosine: $\operatorname{arcosh} = \left(\cosh|_{[0,\infty)}\right)^{-1}$, $D_{\operatorname{arcosh}} = [1, \infty)$
Area hyperbolic tangent: $\operatorname{artanh} = \tanh^{-1}$, $D_{\operatorname{artanh}} = (-1, 1)$
Area hyperbolic cotangent: $\operatorname{arcoth} = \coth^{-1}$, $D_{\operatorname{arcoth}} = (-\infty, -1) \cup (1, \infty)$





Theorem.

1) $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$ $\quad \forall\, x \in \mathbb{R}$
2) $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)$ $\quad \forall\, x \in [1, \infty)$
3) $\operatorname{artanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$ $\quad \forall\, x \in (-1, 1)$
4) $\operatorname{arcoth} x = \frac{1}{2} \ln \frac{x+1}{x-1}$ $\quad \forall\, x \in (-\infty, -1) \cup (1, \infty)$

Proof. 1) $x = \sinh(\operatorname{arsinh} x) = \dfrac{e^{\operatorname{arsinh} x} - e^{-\operatorname{arsinh} x}}{2}$, $x \in \mathbb{R}$

Let $y = e^{\operatorname{arsinh} x} > 0 \implies x = \dfrac{y - \frac{1}{y}}{2} \implies y^2 - 2\,x\,y - 1 = 0$

$\implies y_{1,2} = \dfrac{2\,x \pm \sqrt{4\,x^2+4}}{2} = x \pm \sqrt{x^2+1}$

Since $y > 0 \implies y = x + \sqrt{x^2+1} = e^{\operatorname{arsinh} x} \implies \operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$

2), 3), 4): homework

Derivatives

Theorem.

1) $(\sinh x)' = \cosh x$ $\quad \forall\, x \in \mathbb{R}$
2) $(\cosh x)' = \sinh x$ $\quad \forall\, x \in \mathbb{R}$
5) $(\operatorname{arsinh} x)' = \dfrac{1}{\sqrt{x^2+1}}$ $\quad \forall\, x \in \mathbb{R}$
6) $(\operatorname{arcosh} x)' = \dfrac{1}{\sqrt{x^2-1}}$ $\quad \forall\, x \in (1, \infty)$

3) $(\tanh x)' = \dfrac{1}{\cosh^2 x}$ $\quad \forall x \in (-1, 1)$

4) $(\coth x)' = -\dfrac{1}{\sinh^2 x}$ $\quad \forall x \in \mathbb{R} \setminus \{0\}$

7) $(\operatorname{artanh} x)' = \dfrac{1}{1 - x^2}$ $\quad \forall x \in (-1, 1)$

8) $(\operatorname{arcoth} x)' = \dfrac{1}{1 - x^2}$ $\quad \forall x \in (-\infty, -1) \cup (1, \infty)$

Some proofs.

$$\cosh^2 x - \sinh^2 x = 1 \implies \cosh x = \sqrt{\sinh^2 x + 1}$$
$$\sinh x = \sqrt{\cosh^2 x - 1}$$

5) $(\operatorname{arsinh} x)' = \dfrac{1}{\sinh'(\operatorname{arsinh} x)} = \dfrac{1}{\cosh(\operatorname{arsinh} x)} = \dfrac{1}{\sqrt{\sinh^2(\operatorname{arsinh} x) + 1}} = \dfrac{1}{\sqrt{x^2 + 1}}$

6) $(\operatorname{arcosh} x)' = \dfrac{1}{\cosh'(\operatorname{arcosh} x)} = \dfrac{1}{\sinh(\operatorname{arcosh} x)} = \dfrac{1}{\sqrt{\cosh^2(\operatorname{arcosh} x) - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ $\quad (x > 1)$

Mean value theorems

Local extremum

Definition. The function f has a $\begin{cases} \text{local minimum} \\ \text{local maximum} \end{cases}$ at the point $a \in \operatorname{int} D_f$, if there exists

$\delta > 0$ such that if $x \in (a - \delta, a + \delta)$, then $\begin{cases} f(x) \geq f(a) \\ f(x) \leq f(a) \end{cases}$.

f has a local extremum at a if f has a local minimum or maximum at a.

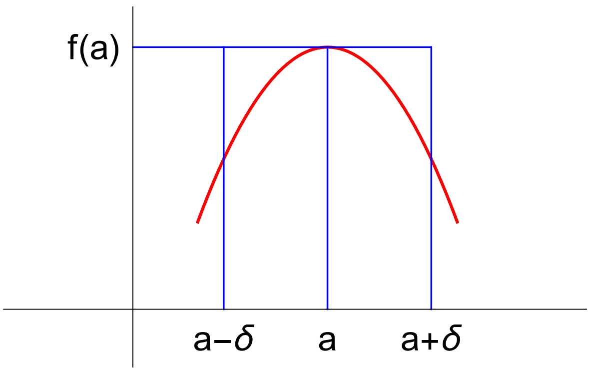
Theorem (Necessary condition for the existence of a local extremum).
If f is differentiable at $a \in \operatorname{int} D_f$ and has a local extremum at a then $f'(a) = 0$.

Proof. Assume that f has a local maximum at $a \in \operatorname{int} D_f$.

If $a - \delta < x < a$ then $f(x) \leq f(a) \implies \dfrac{f(x) - f(a)}{x - a} \geq 0 \implies \mathbf{f'(a)} = f_-'(a) = \lim\limits_{x \to a-} \dfrac{f(x) - f(a)}{x - a} \mathbf{\geq 0}$

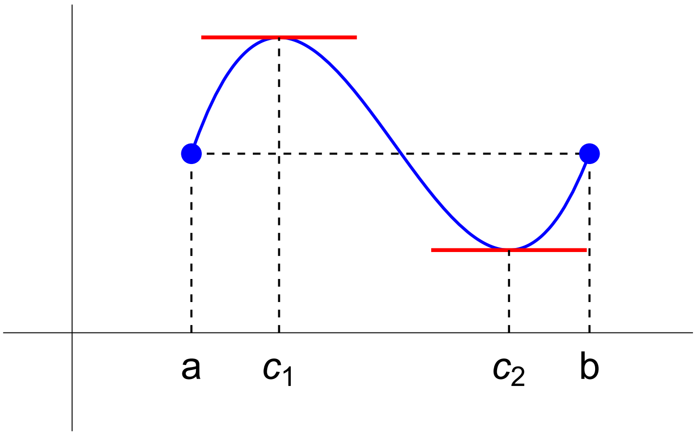
If $a < x < a + \delta$ then $f(x) \leq f(a) \implies \dfrac{f(x) - f(a)}{x - a} \leq 0 \implies \mathbf{f'(a)} = f_+'(a) = \lim\limits_{x \to a+} \dfrac{f(x) - f(a)}{x - a} \mathbf{\leq 0}$

$\implies \mathbf{f'(a) = 0}$.



Rolle's theorem

Theorem (Rolle). Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.



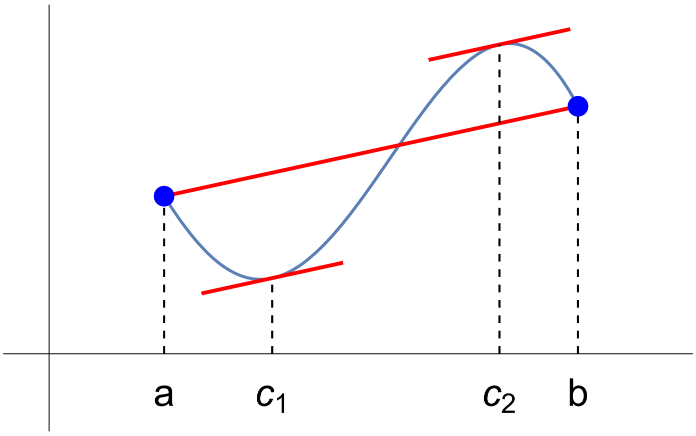
Proof. Since f is continuous on the closed and bounded interval $[a, b]$ then by the Weierstrass extreme value theorem f has a minimum and a maximum on $[a, b]$.

1) If both extreme values are attained at the endpoints, then
$f(x) = f(a) = f(b)$ for all $x \in [a, b] \implies f$ is constant
$\implies f'(c) = 0$ for all $c \in (a, b)$.

2) If the minimum or the maximum is attained at an interior point $c \in (a, b)$,
then f has a local extremum at c, so $f'(c) = 0$.

Lagrange's mean value theorem

Theorem (Lagrange's mean value theorem).
Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b).
Then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Geometrical meaning: There exists a point in the interval where the slope of the tangent line is the same as the slope of the secant line connecting the endpoints of the graph.

Proof. The equation of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ is

$$y = h_{a,b}(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x) = f(x) - h_{a,b}(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a) - f(a)$. Then

1) g is continuous on $[a, b]$
2) g is differentiable on (a, b)
3) $g(a) = g(b) = 0$

$\Longrightarrow$ by Rolle's theorem there exists $c \in (a, b)$ such that $g'(c) = 0$

$\Longrightarrow g'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a} = 0.$

Remark. Rolle's theorem is a special case of this theorem.

Cauchy's mean value theorem

Theorem (Cauchy's mean value theorem).
Assume that $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous on $[a, b]$, differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Then
1) $g(a) \neq g(b)$ and
2) there exists $c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

Proof. 1) If $g(a) = g(b)$ then by Rolle's theorem there exists $c \in (a, b)$ such that $g'(c) = 0$ which is a contradiction.

2) Let $h(x) = (g(b) - g(a)) f(x) - (f(b) - f(a)) g(x)$. Then

- h is continuous on $[a, b]$
- h is differentiable on (a, b)
- $h(a) = h(b) = f(a) g(b) - f(b) g(a)$

$\Longrightarrow$ by Rolle's theorem there exists $c \in (a, b)$ such that

$$h'(c) = (g(b) - g(a)) f'(c) - (f(b) - f(a)) g'(c) = 0 \;\Longrightarrow\; \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Remark. Lagrange's mean value theorem is a special case of this theorem with $g(x) = x$.

Consequence. Assume that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and $f'(x) = 0$ for all $x \in (a, b)$. Then $f(x) = c$ (constant) for all $x \in [a, b]$.

Proof. By Lagrange's mean value theorem for all $[x_1, x_2] \subset [a, b]$ there exists $c \in (x_1, x_2)$

such that $f'(c) = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} = 0 \;\Longrightarrow\; f(x_1) = f(x_2)$ for all $x_1 \neq x_2$

$\Longrightarrow$ f is constant.

Remark. If D_f is not an interval then the statement is not true.

Consequence. Assume that $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous on $[a, b]$, differentiable on (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$.
$\Longrightarrow \exists\, c \in \mathbb{R}$ such that $f(x) = g(x) + c \;\; \forall\, x \in [a, b]$.

Proof. Apply the previous theorem for $f - g$.

Exercises

Exercise. Prove that $f(x) = x^7 + 14\,x - 3$ has exactly one root.

Solution. $f(0) < 0$ and $f(1) > 0 \implies$ by the intermediate value theorem f has a root on $(0, 1)$.
Assume that f has at least two roots: $f(x_1) = f(x_2) = 0$.
Then applying Rolle's theorem on $[x_1, x_2]$: there exists $c \in (x_1, x_2)$ such that $f'(c) = 0$.
However, $f'(x) = 7\,x^6 + 14 > 0$, which is a contradiction.

Exercise. Prove that if $x < y$ then $\arctan y - \arctan x < y - x$.

Solution. $f(x) = \arctan x \implies$ by Lagrange's theorem $\exists\, c \in (x, y)$: $\dfrac{f(y) - f(x)}{y - x} = f'(c)$

$$\implies \frac{\arctan y - \arctan x}{y - x} = \frac{1}{1 + c^2} \leq 1 \implies \arctan y - \arctan x < y - x.$$

Exercise. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Solution. Let $f(x) = \cos x$ and $x > y \implies$ by Lagrange's theorem $\exists\, c \in (y, x)$:

$$\frac{f(x) - f(y)}{x - y} = \frac{\cos x - \cos y}{x - y} = f'(c) = -\sin c$$

$$\implies |\cos x - \cos y| = |(-\sin c)\cdot(x - y)| \leq |x - y|.$$

Remark. From this it follows that $f(x) = \cos x$ is uniformly continuous on $\mathbb{R}$, since for all $\varepsilon > 0$, $\delta = \varepsilon$.