Calculus 1, 14th lecture

Limits of real functions

Definitions

A function $f: A \longrightarrow B$ is a mapping that assigns exactly one element of B to every element from A.
The set A is called the domain of f (notation: D_f or $\mathrm{Dom}(f)$) and the set $f(A) = \{f(x) : x \in A\}$ is called the range of f (notation: R_f or $\mathrm{Ran}(f)$).

A function $f: A \longrightarrow B$ is **one-to one** or **injective** if for all $x, y \in A$: $(f(x) = f(y) \implies x = y)$.
A function $f: A \longrightarrow B$ is **onto** or **surjective** if $f(A) = B$.
A function f is **bijective** if it is injective and surjective.

The function $f: D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is
- even, if $\forall x \in D_f$, $-x \in D_f$ and $f(x) = f(-x)$
- odd, if $\forall x \in D_f$, $-x \in D_f$ and $f(-x) = -f(x)$
- monotonically increasing if $\forall x \in D_f$ $(x \leq y \implies f(x) \leq f(y))$
- monotonically decreasing if $\forall x \in D_f$ $(x \leq y \implies f(x) \geq f(y))$
- strictly monotonically increasing if $\forall x \in D_f$ $(x < y \implies f(x) < f(y))$
- strictly monotonically decreasing if $\forall x \in D_f$ $(x < y \implies f(x) > f(y))$
- periodic with period $p > 0$ if $\forall x \in D_f$, $x + p \in D_f$ and $f(x) = f(x + p)$

Limit at a finite point

Definition. The limit of the function $f: D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is $A \in \mathbb{R}$ if
(1) x_0 is a limit point of D_f $(x \in D_f{}')$
(2) for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that
if $x \in D_f$ and $0 < |x - x_0| < \delta(\varepsilon)$ then $|f(x) - A| < \varepsilon$
Notation: $\lim\limits_{x \to x_0} f(x) = A$

Example 1. $\lim\limits_{x \to -2} \dfrac{8 - 2x^2}{x + 2} = 8$, since if $\varepsilon > 0$, then

$$|f(x) - A| = \left| \frac{8 - 2x^2}{x + 2} - 8 \right| = \left| \frac{2 \cdot (4 - x^2)}{x + 2} - 8 \right| = |2 \cdot (2 - x) - 8| =$$

$$= |-2x - 4| = 2|x + 2| < \varepsilon, \text{ if } |x + 2| < \frac{\varepsilon}{2}$$

$\implies$ with the choice $\delta(\varepsilon) = \dfrac{\varepsilon}{2}$ the definition holds. (Here $-2 \notin D_f$.)

Example 2. $\lim\limits_{x \to -3} \sqrt{1 - 5x} = 4$, since if $\varepsilon > 0$, then

$$|f(x) - A| = \left| \sqrt{1 - 5x} - 4 \right| = \left| \frac{1 - 5x - 16}{\sqrt{1 - 5x} + 4} \right| = \frac{5|x + 3|}{\sqrt{1 - 5x} + 4} \leq \frac{5|x + 3|}{0 + 4} < \varepsilon,$$

if $|x + 3| < \dfrac{4\varepsilon}{5}$ $\implies$ with the choice $\delta(\varepsilon) = \dfrac{4\varepsilon}{5}$ the definition holds.

Definition. Suppose $f: D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a function and $x_0 \in D_f\,'$. Then $\lim\limits_{x\to x_0} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if

for all $P > 0$ there exists $\delta(P) > 0$ such that

if $x \in D_f$ and $0 < |x - x_0| < \delta(\varepsilon)$ then $\begin{cases} f(x) > P \\ f(x) < -P \end{cases}.$

Example 3. $\lim\limits_{x\to 2} \dfrac{1}{(x-2)^2} = \infty$, since if $P > 0$, then $f(x) = \dfrac{1}{(x-2)^2} > P \Longleftrightarrow 0 < \left| x - 2 \right| < \dfrac{1}{\sqrt{P}}$

$\Longrightarrow$ with the choice $\delta(P) = \dfrac{1}{\sqrt{P}}$ the definition holds.

Limit at ∞ and $-\infty$

Definitions.

(1) $\lim\limits_{x\to\infty} f(x) = A \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists $P(\varepsilon) > 0$ such that if $x > P(\varepsilon)$ then $|f(x) - A| < \varepsilon$.

(2) $\lim\limits_{x\to\infty} f(x) = \infty$ if for all $K > 0$ there exists $P(K) > 0$ such that if $x > P(K)$ then $f(x) > K$.

(3) $\lim\limits_{x\to\infty} f(x) = -\infty$ if for all $K > 0$ there exists $P(K) > 0$ such that if $x > P(K)$ then $f(x) < -K$.

Remark. If f is a sequence, that is, $D_f = \mathbb{N}^+$, then the only accumulation point of D_f is ∞, so can we investigate the limit only here.

Definitions.

(1) $\lim\limits_{x\to-\infty} f(x) = A \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists $P(\varepsilon) > 0$ such that if $x < -P(\varepsilon)$ then $|f(x) - A| < \varepsilon$.

(2) $\lim\limits_{x\to-\infty} f(x) = \infty$ if for all $K > 0$ there exists $P(K) > 0$ such that if $x < -P(K)$ then $f(x) > K$.

(3) $\lim\limits_{x\to-\infty} f(x) = -\infty$ if for all $K > 0$ there exists $P(K) > 0$ such that if $x < -P(K)$ then $f(x) < -K$.

Summary

The above definitions of the limit can be summarized as follows.

Theorem. Assume that $a \in \overline{\mathbb{R}}$ is a limit point of D_f and $b \in \overline{\mathbb{R}}$. Then $\lim\limits_{x\to a} f(x) = b$ if and only if

for any neighbourhood J of b there exists a neighbourhood I of a such that

if $x \in I \cap D_f$ and $x \neq a$ then $f(x) \in J$.

The sequential criterion for a limit of a function

In the syllabus it is called transference principle.

Theorem. Suppose $f: D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a function, $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, and $a \in D_f\,'$.

Then the following two statements are equivalent.

(1) $\lim\limits_{x\to a} f(x) = b$

(2) For all sequences $(x_n) \subset D_f \setminus \{a\}$ for which $x_n \longrightarrow a$, $\lim\limits_{n\to\infty} f(x_n) = b$.

Proof. We prove it for $a, b \in \mathbb{R}$.

(1) $\Longrightarrow$ (2): Assume that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $0 < |x - a| < \delta(\varepsilon)$

then $|f(x) - b| < \varepsilon$.

Let (x_n) be a sequence for which $x_n \in D_f \setminus \{a\}$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow a$.

Then for $\delta(\varepsilon)>0$ there exists a threshold index $N(\delta(\varepsilon))\in\mathbb{N}$ such that if $n>N(\delta(\varepsilon))$ then $|x_n-a|<\delta(\varepsilon)$.

Thus for all $n>N(\delta(\varepsilon))$, $|f(x_n)-b|<\varepsilon$ also holds, so $f(x_n)\longrightarrow b$.

(2) $\Longrightarrow$ (1): Indirectly, assume that (2) holds but $\lim_{x\to a} f(x)\neq b$, that is,

there exists $\varepsilon>0$ such that for all $\delta>0$ there exists $x\in D_f$ for which $0<|x-a|<\delta$ and $|f(x)-b|\geq\varepsilon$.

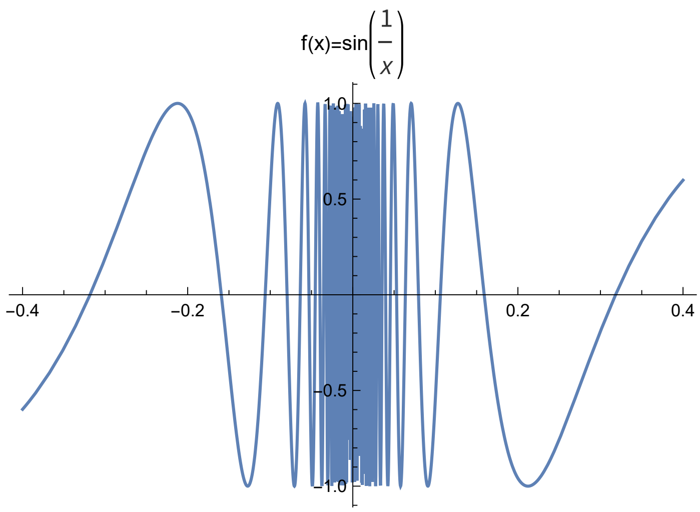
Let $\delta_n=\frac{1}{n}>0$ for all $n\in\mathbb{N}^+$. Then for δ_n there exists $x_n\in D_f$ such that $0<|x_n-a|<\delta$ and $|f(x_n)-b|\geq\varepsilon$.

It means that $x_n\longrightarrow a$, but $\lim_{n\to\infty} f(x_n)\neq b$, which is a contradiction, so $\lim_{x\to a} f(x)=b$.

Example. The limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist. Let $x_n=\frac{1}{n\pi}\longrightarrow 0$ and $y_n=\frac{1}{\frac{\pi}{2}+2n\pi}\longrightarrow 0$. Then

$$\lim_{n\to\infty}\sin\left(\frac{1}{x_n}\right)=\lim_{x\to\infty}\sin(n\pi)=0 \text{ and}$$

$$\lim_{x\to\infty}\sin\left(\frac{1}{y_n}\right)=\lim_{x\to\infty}\sin\left(\frac{\pi}{2}+2n\pi\right)=1 \text{ and } 0\neq 1.$$



Consequences

Theorem. Suppose $x_0\in\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$ is a limit point of $D_f\cap D_g$ and $\lim_{x\to x_0} f(x)=A\in\mathbb{R}$, $\lim_{x\to x_0} g(x)=B\in\mathbb{R}$, $c\in\mathbb{R}$. Then

(1) $\lim_{x\to x_0}(c\,f)(x)=c\cdot A$

(2) $\lim_{x\to x_0}(f\pm g)(x)=A\pm B$

(3) $\lim_{x\to x_0}(f\cdot g)(x)=A\cdot B$

(4) $\lim_{x\to x_0}\left(\frac{f}{g}\right)(x)=\frac{A}{B}$

(5) If $\lim_{x\to x_0} f(x)=0$ and g is bounded in a neighbourhood of x_0 then $\lim_{x\to x_0}(f\,g)(x)=0$.

Remark. The statements (1)-(4) are also true if $A, B \in \overline{\mathbb{R}}$ and the corresponding operations are defined in $\overline{\mathbb{R}}$.

Theorem. Suppose $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is a limit point of $D_f \cap D_g$ and

$$\lim_{x \to x_0} f(x) = A \in \overline{\mathbb{R}}, \ \lim_{x \to x_0} g(x) = B \in \overline{\mathbb{R}}.$$

If $f(x) \leq g(x)$ for all $x \in D_f \cap D_g$ then $A \leq B$.

Theorem (Sandwich theorem for limits). Suppose that

(1) $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is a limit point of $D_f \cap D_g \cap D_h$,

(2) $f(x) \leq g(x) \leq h(x)$ for all x in a neighbourhood of x_0 and

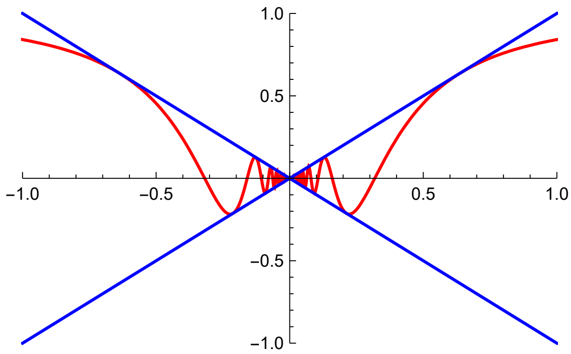
(3) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = b \in \overline{\mathbb{R}}$.

Then $\lim_{x \to x_0} g(x) = b$.

Remark. If $b = \pm\infty$ then only one estimation is enough.

Example. $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$, since $-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$.

Or, $x \longrightarrow 0$ and $\sin\left(\frac{1}{x}\right)$ is bounded, so the product also tends to 0.



One-sided limits

Notation. The $\begin{cases} \text{right hand limit} \\ \text{left hand limit} \end{cases}$ of f at x_0 is denoted as $\begin{cases} \lim_{x \to x_0 +} f(x) = \lim_{x \to x_0 + 0} f(x) = f(x_0 + 0) \\ \lim_{x \to x_0 -} f(x) = \lim_{x \to x_0 - 0} f(x) = f(x_0 - 0) \end{cases}$.

Definition. Suppose $x_0 \in \mathbb{R}$ is a limit point of $\begin{cases} D_f \cap [x_0, \infty) \\ D_f \cap (-\infty, x_0] \end{cases}$. Then

(1) $\begin{cases} \lim_{x \to x_0 +} f(x) = A \\ \lim_{x \to x_0 -} f(x) = A \end{cases}$ if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\begin{cases} x_0 < x < x_0 + \delta(\varepsilon) \\ x_0 - \delta(\varepsilon) < x < x_0 \end{cases}$

then $|f(x) - A| < \varepsilon$.

(2) $\lim_{x\to x_0+} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if for all $K>0$ there exists $\delta(K)>0$ such that if $x_0 < x < x_0 + \delta(K)$

then $\begin{cases} f(x) > P(K) \\ f(x) < -P(K) \end{cases}$.

(3) $\lim_{x\to x_0-} f(x) = \begin{cases} \infty \\ -\infty \end{cases}$ if for all $K>0$ there exists $P(K)>0$ such that if $x_0 - \delta(K) < x < x_0$

then $\begin{cases} f(x) > P(K) \\ f(x) < -P(K) \end{cases}$.

Definition. Let $f : X \longrightarrow Y$ be a function and $A \subset X$. The **restriction of *f* to *A*** is the function

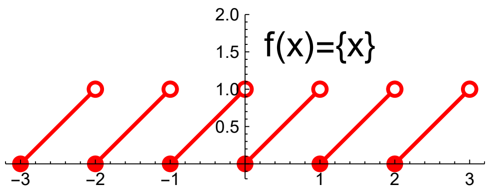
$$f\,|_A : A \longrightarrow Y,\quad f\,|_A(x) = f(x).$$

Remarks. 1) $\lim_{x\to x_0+} f(x) = \lim_{x\to x_0} f\,|_{D_f\cap[x_0,\infty)}(x)$, $\lim_{x\to x_0-} f(x) = \lim_{x\to x_0} f\,|_{D_f\cap(-\infty,x_0]}(x)$

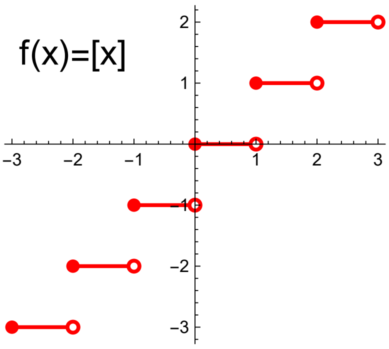
2) Suppose $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ and x_0 is a limit point of D_f $(x \in D_f')$. Then $\lim_{x\to x_0} f(x)$ exists if and only if $\lim_{x\to x_0+} f(x)$ and $\lim_{x\to x_0-} f(x)$ exist and $\lim_{x\to x_0+} f(x) = \lim_{x\to x_0-} f(x)$.

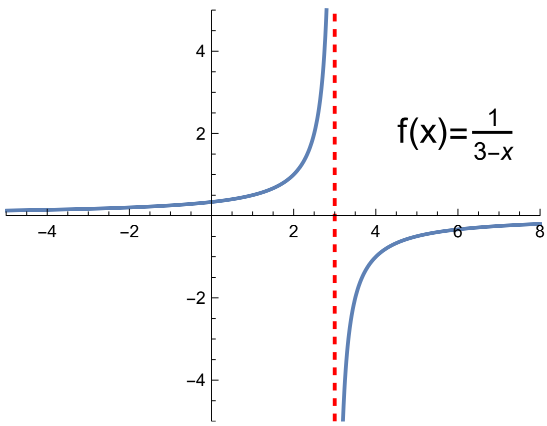
Examples. 1) $\lim_{x\to 1-0} [x] = 0$, $\lim_{x\to 1+0} [x] = 1$

2) $\lim_{x\to 1-0} \{x\} = 1$, $\lim_{x\to 1+0} \{x\} = 0$



3) $\lim_{x\to 3+0} \dfrac{1}{3-x} = -\infty$, $\lim_{x\to 3-0} \dfrac{1}{3-x} = +\infty$





Continuity

Definition. The function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is $\begin{cases} \text{continuous} \\ \text{continuous from the right} \\ \text{continuous from the left} \end{cases}$ at the point $x_0 \in D_f$ if

for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in D_f$ and $\begin{cases} |\,x - x_0\,| < \delta(\varepsilon) \\ x_0 - \delta(\varepsilon) < x \le x_0 \\ x_0 \le x < x_0 + \delta(\varepsilon) \end{cases}$

then $|\,f(x) - f(x_0)\,| < \varepsilon$.

Remarks. 1) f is continuous at $x_0 \in D_f \iff$ for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in (B(x_0, \delta) \cap D_f$ then $f(x) \in B(f(x_0), \varepsilon)$.

2) f is $\begin{cases} \text{continuous from the right} \\ \text{continuous from the left} \end{cases}$ at $x_0 \in D_f \iff \begin{cases} f\,|_{D_f \cap [x_0, \infty)} \\ f\,|_{D_f \cap (-\infty, x_0]} \end{cases}$ is continuous at x_0.

3) f is continuous at $x_0 \in D_f \iff f$ is continuous at x_0 from the right and from the left.

Definition. f is continuous if f is continuous for all $x \in D_f$.

Notation. If $A \subset \mathbb{R}$ then $C(A, \mathbb{R})$ or $C(A)$ denotes the set of continuous functions $f : A \longrightarrow \mathbb{R}$.
For example, $f \in C([a, b])$ means that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous.

Theorem. Suppose $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ and $x_0 \in D_f \cap D_f{}'$. Then f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x)$ exists and $\lim_{x \to x_0} f(x) = f(x_0)$.

The sequential criterion for continuity

Theorem: The function $f : D_f \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $x_0 \in D_f$ if and only if for all sequences $(x_n) \subset D_f$ for which $x_n \longrightarrow x_0$, $\lim_{n \to \infty} f(x_n) = f(x_0)$.

Consequences

Theorem. If f and g are continuous at $x_0 \in D_f \cap D_g$ then $c\,f$, $f \pm g$ and $f\,g$ is continuous at x_0 $(c \in \mathbb{R})$.
If $g(x_0) \neq 0$ then $\frac{f}{g}$ is also continuous at x_0.

Theorem (Sandwich theorem for continuity): Suppose that
(1) there exists $\delta > 0$ such that $I = (x_0 - \delta, x_0 + \delta) \subset D_f \cap D_g \cap D_h$
(2) f and h are continuous at x_0
(3) $f(x_0) = h(x_0)$
(4) $f(x) \leq g(x) \leq h(x)$ for all $x \in I$
Then g is continuous at x_0.

Definition. The composition of the functions f and g is $(f \circ g)(x) = f(g(x))$ whose domain is
$D_{f \circ g} = \{x \in D_g : g(x) \in D_f\}$.

Theorem. If g is continuous at $x_0 \in D_g$ and f is continuous at $g(x_0) \in D_f$ then $f \circ g$ is continuous at x_0.

Theorem (Limit of a composition). Let a be a limit point of $D_{f \circ g}$ for which $\lim_{x \to a} g(x) = b$.

Assume that
(1) $b \in D_f$, f is continuous at b and $f(b) = c$ **or**
(2) $b \in D_f{}' \setminus D_f$ and $\lim_{x \to b} f(x) = c$ **or**
(3) g is injective, $b \in D_f{}'$ and $\lim_{x \to b} f(x) = c$.

Then $\lim_{x \to a} (f \circ g)(x) = c$.

Examples

1) The constant function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = c$ is continuous for all $x_0 \in \mathbb{R}$.
Let $\varepsilon > 0$, then with any $\delta > 0$ if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| = |c - c| = 0 < \varepsilon$.

2) $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)=x$ is continuous for all $x_0\in\mathbb{R}$.

Let $\varepsilon>0$, then with $\delta(\varepsilon)=\varepsilon$ if $|x-x_0|<\delta(\varepsilon)=\varepsilon$, then $|f(x)-f(x_0)|=|x-x_0|=0<\varepsilon$.

3) $f:\mathbb{R}\longrightarrow\mathbb{R},\ f(x)=x^n$ is continuous for all $x_0\in\mathbb{R}$, $n\in\mathbb{N}$, since

$f(x)=x^n=x\cdot x\cdot\ldots\cdot x\longrightarrow x_0\cdot x_0\cdot\ldots\cdot x_0=x_0^n=f(x_0)$

4) Polynomials ($P_n(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0,\ \ a_i\in\mathbb{R}$) are continuous for all $x_0\in\mathbb{R}$.

5) The Dirichlet function $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$ is not continuous for all $x\in\mathbb{R}$.

If $x_0\in\mathbb{Q}$, then let $x_n\in\mathbb{R}\setminus\mathbb{Q}\ \ \forall n$ such that $x_n\longrightarrow x_0$. Then $f(x_n)=0\longrightarrow 0\neq 1=f(x_0)$.

If $x_0\in\mathbb{R}\setminus\mathbb{Q}$, then let $x_n\in\mathbb{Q}\ \ \forall n$ such that $x_n\longrightarrow x_0$. Then $f(x_n)=1\longrightarrow 1\neq 0=f(x_0)$.

6) $f(x)=\sin x$ and $g(x)=\cos x$ are continuous for all $x\in\mathbb{R}$.

(i) $\begin{cases}0<\sin x<x & \text{if } x>0\\ x<\sin x<0 & \text{if } x<0\end{cases}\Longrightarrow\ \lim\limits_{x\to 0}\sin x=0.$

(ii) $\cos x=1-\sin^2\dfrac{x}{2}\ \Longrightarrow\ \lim\limits_{x\to 0}\cos x=1$. Therefore

(iii) $\lim\limits_{x\to x_0}\sin x=\sin x_0\lim\limits_{x\to x_0}\cos(x-x_0)+\cos x_0\lim\limits_{x\to x_0}\sin(x-x_0)=\sin x_0$

(iv) $\lim\limits_{x\to x_0}\cos x=\cos x_0\lim\limits_{x\to x_0}\cos(x-x_0)-\sin x_0\lim\limits_{x\to x_0}\sin(x-x_0)=\cos x_0$

7) $f(x)=\begin{cases}x\sin\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$ is continuous for all $x\in\mathbb{R}$.

Example

Theorem. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

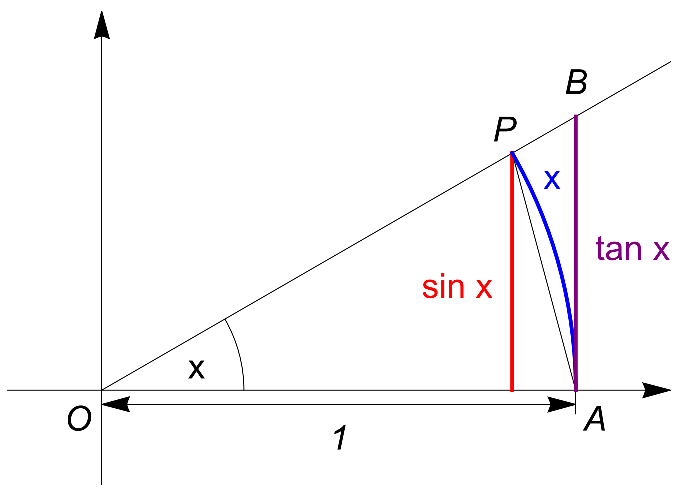
Proof. Since $f(x)=\dfrac{\sin x}{x}$ is even, it is enough to consider the right-hand limit $\lim\limits_{x\to 0+}\dfrac{\sin x}{x}$.

Let $0<x<\dfrac{\pi}{2}$.

The area of the POA triangle is $T_1=\dfrac{1\cdot\sin x}{2}$.

The area of the POA circular sector is $T_2=\dfrac{1^2\cdot x}{2}$.

The area of the OAB triangle is $T_3=\dfrac{1\cdot\tan x}{2}$.



Obviously $T_1 < T_2 < T_3$, that is, $\frac{1 \cdot \sin x}{2} < \frac{1^2 \cdot x}{2} < \frac{1 \cdot \tan x}{2}$.

Multiplying both sides by $\frac{2}{\sin x} > 0$: $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$.

Since $\lim\limits_{x \to 0+} \frac{1}{\cos x} = 1$ then $\lim\limits_{x \to 0+} \frac{x}{\sin x} = 1 \implies \lim\limits_{x \to 0+} \frac{\sin x}{x} = 1 = \lim\limits_{x \to 0-} \frac{\sin x}{x}$

Remark. If $0 < x < \frac{\pi}{2}$, then $\sin x < x \implies |\sin x| \leq |x| \quad \forall x \in \mathbb{R}$.

Types of discontinuities

Definition. The function f is **discontinuous** at $x_0 \in D_f$ or f has a discontinuity at $x_0 \in D_f$ if f is not continuous at x_0.

Classification of discontinuities:

1) Discontinuity of the first kind:

a) f has a **removable discontinuity** at x_0 if $\exists \lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$.

b) f has a **jump discontinuity** at x_0 if $\exists \lim\limits_{x \to x_0-} f(x) \in \mathbb{R}$ and $\exists \lim\limits_{x \to x_0+} f(x) \in \mathbb{R}$

but $\lim\limits_{x \to x_0-} f(x) \neq \lim\limits_{x \to x_0+} f(x)$.

2) Discontinuity of the second kind:

f has an **essential discontinuity** or a discontinuity of the second kind at x_0 if f has a discontinuity at x_0 but not of the first kind.

Examples. 1. a) $f(x) = \frac{x^2 - 1}{x - 1}$ has a removable discontinuity at $x_0 = 1$.

1. b) $f(x) = [x]$ has a jump discontinuity for all $x \in \mathbb{Z}$.

2. $f_1(x) = \frac{1}{x}$, $f_2(x) = \frac{1}{x^2}$ and $f_3 = \sin \frac{1}{x}$ have an essential discontinuity at $x_0 = 0$.

The Dirichlet function has essential discontinuities for all $x \in \mathbb{R}$.