Calculus 1, 8th and 9th lecture

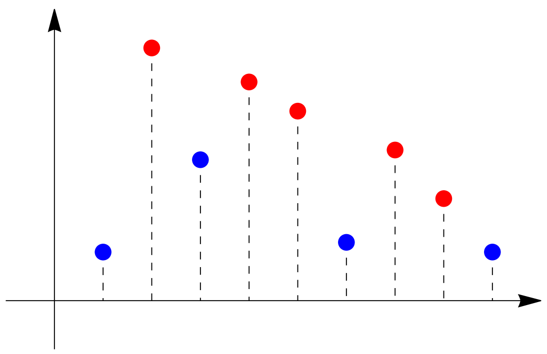
Bolzano-Weierstrass theorem

Theorem: Every sequence has a monotonic subsequence.

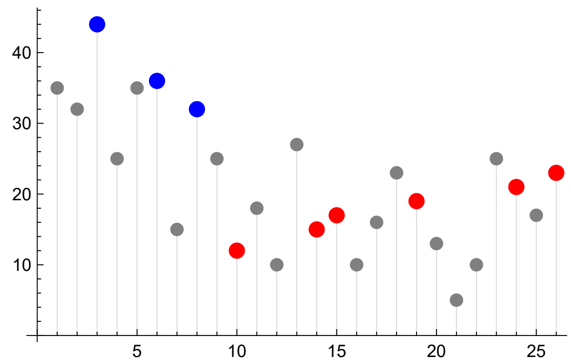
Proof. First we introduce the following concept: a_k is called a **peak element** if $a_n \le a_k$ for all $n > k$. Then two cases are possible.
Case 1: There are infinitely many peak elements. If $n_1 < n_2 < n_3 < \ldots$ are indexes for which $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ are peak elements, then the sequence $a_{n_1}, a_{n_2}, a_{n_3}$ is monotonically decreasing.

Case 2: There are finitely many peak elements (or none). It means that there exists an index n_0 such that for all $n \ge n_0$, a_n is not a peak element.
Since a_{n_0} is not a peak element, there exists $n_1 > n_0$ such that $a_{n_1} > a_{n_0}$.
Since a_{n_1} is not a peak element, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$, etc.
In this case the sequence $a_{n_0}, a_{n_1}, a_{n_2}$ is strictly monotonic increasing.

Case 1:



Case 2:



Theorem (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Proof: Because of the previous theorem there exists a monotonic subsequence and since it is bounded then it is convergent.

Remark. The Bolzano-Weierstrass theorem is not true in the set of rational numbers.
Let $(b_n) = (1, 1.4, 1.41, 1.414, \ldots) \longrightarrow \sqrt{2} \notin \mathbb{Q}$, then $b_n \in \mathbb{Q}$ and $b_n \in [1, 2]$ for all n, that is, (b_n) is bounded.
Each subsequence of (b_n) converges to $\sqrt{2}$, so (b_n) does not have a subsequence converging to a rational number.

Cauchy sequences

Definition. (a_n) is a **Cauchy sequence** if for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $n, m > N$ then $|a_n - a_m| < \varepsilon$.

Statement: If (a_n) is a Cauchy sequence, then it is bounded, since for all $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\min\{a_{N+1} - \varepsilon, a_1, \ldots, a_N\} \leq a_n \leq \max\{a_{N+1} + \varepsilon, a_1, \ldots, a_N\}.$$

Theorem. (a_n) is convergent if and only if it is a Cauchy sequence.

Proof. a) Let $\varepsilon > 0$ be fixed. If $\lim_{n\to\infty} a_n = A$, then for $\frac{\varepsilon}{2}$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $\left| a_n - A \right| < \frac{\varepsilon}{2}$.

So if $n, m > N$ then $| a_n - a_m | = \left| a_n - A + A - a_m \right| \leq \left| a_n - A \right| + \left| A - a_m \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

b) If (a_n) is a Cauchy sequence then it is bounded. Define $c_n = \inf\{a_n, a_{n+1}, \ldots\}$ and $d_n = \sup\{a_n, a_{n+1}, \ldots\}$.

Then $c_n \leq c_{n+1} \leq d_{n+1} \leq d_n$, so by the Cantor-axiom $\bigcap_{n=1}^{\infty} [c_n, d_n] \neq \emptyset$. Since for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $| c_n - d_n | < \varepsilon$, it means that the intersection has only one element A, which is the limit of the sequence
$(| A - a_n | < \max\{| c_n - a_n |, | d_n - a_n |\} < \varepsilon)$.

Remark. The theorem expresses the fact that the terms of a convergent sequence are also arbitrarily close to each other if their indexes are large enough. The theorem can be used to prove convergence even if the limit is not known.

Example. $a_n = (-1)^n$ is not convergent, since $| a_n - a_{n+1} | = \left| (-1)^n - (-1)^{n+1} \right| = 2 \geq \varepsilon$ if $\varepsilon \leq 2$.

Remark. A Cauchy sequence is not necessarily convergent in the set of rational numbers.
For example $(a_n) = (1, 1.4, 1.41, 1.414, \ldots) \longrightarrow \sqrt{2} \notin \mathbb{Q}$.
(a_n) is a Cauchy sequence, since $\left| a_{n+k} - a_n \right| < 10^{-N}$ if $n > N$ and $k \in \mathbb{N}$ is arbitrary, but the limit of (a_n) is not rational.

An important example

Let $s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Prove that $\lim_{n\to\infty} s_n = \infty$.

Solution. Let $\varepsilon \leq \frac{1}{2}$ and $m = 2n$. Then with

$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ and $s_m = s_{2n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) + \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$,
we get that

$$| s_m - s_n | = | s_{2n} - s_n | = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} = n \cdot \frac{1}{2n} = \frac{1}{2} \geq \varepsilon,$$

so (s_n) is not a Cauchy sequence. Since (s_n) is monotonically increasing, it follows that $s_n \longrightarrow \infty$.

Limit points or accumulation points of a sequence

Definition. For any $P \in \mathbb{R}$, the interval (P, ∞) is called a neighbourhood of $+\infty$ and the interval $(-\infty, P)$ is called a neighbourhood of $-\infty$.

Definition. $A \in \mathbb{R} \cup \{\infty, -\infty\}$ is called a **limit point** or **accumulation point** of (a_n) if any neighbourhood of A contains infinitely many terms of (a_n). Or equivalently there exists a subsequence (a_{n_k}) such that $a_{n_k} \xrightarrow{n\to\infty} A$.

Theorem. Every sequence has at least one limit point.

Proof. We proved that every sequence has a monotonic subsequence.
If it is bounded, then it has a finite limit, so it is a limit point of the sequence.
If the subsequence is not bounded, then it tends to ∞ or $-\infty$, so ∞ or $-\infty$ is a limit point of the sequence.

Definition. If the set of limit points of (a_n) is bounded above, then its supremum is called the **limes superior** of (a_n) (notation: $\limsup a_n$), and if it is bounded below, then we call the infimum of the set the **limes inferior** of (a_n) (notation: $\liminf a_n$).
If (a_n) is not bounded above, we define $\limsup a_n = \infty$ and if (a_n) is not bounded below, then we define $\liminf a_n = -\infty$.

Theorem. (a_n) is convergent if and only if $\limsup a_n = \liminf a_n = A \in \mathbb{R}$.

Proof. 1) If (a_n) is convergent, then all of its subsequences tend to the same limit as (a_n), so the only element of the set of the limit points will be the limsup and the liminf of the sequence.
2) Let $\limsup a_n = \liminf a_n = A$ and let $\varepsilon > 0$ be fixed. If we assume indirectly that $\lim\limits_{n\to\infty} a_n \neq A$ then it means that there are infinitely many terms $n_1 < n_2 < \ldots \in \mathbb{N}$ such that $|a_n - A| \geq \varepsilon$. Then (a_{n_k}) has a limit point which differs from A, so we arrived at a contradiction.

Numerical series

Definition

Definition. Suppose (a_n) is a sequence and define the sequence of **partial sums** as $s_n = \sum\limits_{k=1}^{n} a_k$. If s_n is convergent, then the **numerical series** $\sum\limits_{n=1}^{\infty} a_n$ is convergent, and its sum is $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k = \lim\limits_{n\to\infty} s_n$.

The geometric series

Theorem. $1 + q + q^2 + \ldots = \sum_{n=0}^{\infty} q^n = \dfrac{1}{1-q}$ if $\left| q \right| < 1$ and the series is divergent otherwise.

Proof. If $a_n = q^n$ then $s_n = \sum_{k=0}^{n} q^k = \begin{cases} \dfrac{q^{n+1}-1}{q-1} & \text{if } q \neq 1 \\ n+1 & \text{if } q = 1 \end{cases}$

If $q = 1$ then $\lim_{n\to\infty} s_n = \infty$.

If $q > 1$ then $\lim_{n\to\infty} s_n = \infty$, since $\lim_{n\to\infty} q^{n+1} = \infty$.

If $-1 < q < 1$ then $\lim_{n\to\infty} s_n = \dfrac{1}{1-q}$, since $\lim_{n\to\infty} q^{n+1} = 0$.

If $q \le -1$ then $\lim_{n\to\infty} s_n$ does not exist, since $\lim_{n\to\infty} q^n$ does not exist.

Similarly, $\sum_{n=0}^{\infty} a \cdot q^n = \dfrac{a}{1-q}$, $\sum_{n=k}^{\infty} a \cdot q^n = \dfrac{a \cdot q^k}{1-q}$ if $| q | < 1$.

A telescoping series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} \right) =$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \ldots + \frac{1}{n} - \frac{1}{n+1} \right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right) = 1$$

Cauchy criterion

Theorem: The numerical series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m > n > N$ then $| s_m - s_n | = \sum_{k=n+1}^{m} a_k = | a_{n+1} + a_{n+2} + \ldots + a_m | < \varepsilon$.

Proof: It is trivially true, since the Cauchy criterion for number sequences can be applied for (s_n).

The *n*th term test

Theorem: If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

1st proof: Apply the Cauchy criterion with the choice $m = n + 1$. Then

$$| s_{n+1} - s_n | = | a_{n+1} | < \varepsilon \text{ if } n > N(\varepsilon), \text{ so } \lim_{n\to\infty} a_n = 0.$$

2nd proof: Let $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, then $s_n = s_{n-1} + a_n \implies a_n = s_n - s_{n-1} \longrightarrow s - s = 0$.

Remark. The theorem can also be stated in the following form:

If $\lim_{n\to\infty} a_n \neq 0$ or if the limit doesn't exist then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. The condition $\lim_{n\to\infty} a_n = 0$ is necessary but not sufficient for the convergence of $\sum_{n=1}^{\infty} a_n$, as the following example shows.

The harmonic series

Theorem. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof.
$$s_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{n-1}+1} + \ldots + \frac{1}{2^n}\right) \geq$$
$$\geq 1 + \frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + \ldots + 2^{n-1}\cdot\frac{1}{2^n} = 1 + \frac{n}{2} \xrightarrow{n\to\infty} \infty.$$

Remark. The name of the harmonic series comes from the fact that for all $n \geq 2$, a_n is the harmonic mean of a_{n-1} and a_{n+1}, that is,

$$a_n = \frac{2}{\frac{1}{a_{n-1}} + \frac{1}{a_{n+1}}} = \frac{2}{\frac{1}{\frac{1}{n-1}} + \frac{1}{\frac{1}{n+1}}} = \frac{2}{(n-1)+(n+1)} = \frac{1}{n}.$$

The divergence of the series is very slow, for example

$$\sum_{n=1}^{100} \frac{1}{n} \approx 5.18738, \quad \sum_{n=1}^{10^4} \frac{1}{n} \approx 9.78761, \quad \sum_{n=1}^{10^5} \frac{1}{n} \approx 12.0901, \quad \sum_{n=1}^{10^6} \frac{1}{n} \approx 14.3927$$

Sum and constant multiple

Theorem: Assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, $\sum_{n=1}^{\infty} d_n$ is divergent, and $c \in \mathbb{R}\setminus\{0\}$. Then

(1) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(2) $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$

(3) $\sum_{n=1}^{\infty} (a_n + d_n)$ is divergent

(4) $\sum_{n=1}^{\infty} c\, d_n$ is divergent

Proof. All statements follow from the properties of the sequences.

Series with nonnegative terms

Theorem. A series with nonnegative terms converges if and only if its partial sums form a bounded sequence.

Proof. If $a_n \geq 0$ for all $n \in \mathbb{N}$ then $s_{n+1} = a_{n+1} + s_n \geq s_n$ for all $n \in \mathbb{N}$, so (s_n) is monotonically increasing.

If $\sum_{n=1}^{\infty} a_n$ converges, then (s_n) converges $\Longrightarrow$ (s_n) is bounded.

If (s_n) is bounded, then (s_n) converges since it is monotonically increasing.

Cauchy Condensation Test

Theorem. Suppose $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges

if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2 a_2 + 4 a_4 + 8 a_8 + \ldots$ converges.

Proof. Let $s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$ and

$$\sigma_n = a_1 + 2 a_2 + 4 a_4 + 8 a_8 + \ldots + 2^n a_{2^n} = \sum_{k=1}^{n} 2^k a_{2^k}$$

1) (s_n) is monotonically increasing, since the terms of (a_n) are nonnegative and $n \leq 2^n - 1$ for all $n \in \mathbb{N}^+$ so $s_n \leq s_{2^n-1}$. Then

$$\begin{aligned} s_n \leq s_{2^n-1} &= \mathbf{a_1} + (\mathbf{a_2} + \mathbf{a_3}) + (\mathbf{a_4} + \mathbf{a_5} + \mathbf{a_6} + \mathbf{a_7}) + \ldots + (a_{2^{n-1}} + \ldots + a_{2^n-1}) \leq \\ &\leq \mathbf{a_1} + (\mathbf{a_2} + \mathbf{a_2}) + (\mathbf{a_4} + \mathbf{a_4} + \mathbf{a_4} + \mathbf{a_4}) + \ldots + (a_{2^{n-1}} + \ldots + a_{2^{n-1}}) = \\ &= \mathbf{a_1} + \mathbf{2\, a_2} + \mathbf{4\, a_4} + \ldots + 2^{n-1} a_{2^{n-1}} = \\ &= \frac{1}{2} (a_1 + 2 a_2 + 4 a_4 + 8 a_8 + \ldots + 2^n a_{2^n}) = \sigma_{n-1} \end{aligned}$$

Assume that $\sum_{k=1}^{n} 2^k a_{2^k}$ is convergent $\Longrightarrow$ (σ_n) is convergent, so it is bounded $\Longrightarrow$ (s_n) is bounded above since $s_n \leq s_{2^n-1} \leq \sigma_{n-1}$ $\Longrightarrow$ (s_n) is convergent since it is monotonically increasing.

2)
$$\begin{aligned} s_{2^n} &= \mathbf{a_1} + \mathbf{a_2} + (\mathbf{a_3} + \mathbf{a_4}) + (\mathbf{a_5} + \mathbf{a_6} + \mathbf{a_7} + \mathbf{a_8}) + \ldots + (a_{2^{n-1}+1} + \ldots + a_{2^n}) \geq \\ &\geq \mathbf{\frac{1}{2}}\, \mathbf{a_1} + \mathbf{a_2} + (\mathbf{a_4} + \mathbf{a_4}) + (\mathbf{a_8} + \mathbf{a_8} + \mathbf{a_8} + \mathbf{a_8}) + \ldots + (a_{2^n} + \ldots + a_{2^n}) = \\ &= \mathbf{\frac{1}{2}}\, \mathbf{a_1} + \mathbf{a_2} + \mathbf{2\, a_4} + \mathbf{4\, a_8} + \ldots + 2^{n-1} a_{2^n} = \frac{1}{2} \sigma_n \end{aligned}$$

Assume that $\sum_{n=1}^{\infty} a_n$ is convergent $\Longrightarrow$ (s_n) is convergent, so it is bounded $\Longrightarrow$ (σ_n) is bounded above since $\frac{1}{2} \sigma_n \leq s_{2^n}$ $\Longrightarrow$ (σ_n) is convergent since it is monotonically increasing $\Longrightarrow$ $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent.

The p-series (or hyperharmonic series)

Theorem. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. If $p \leq 0$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{|p|} \neq 0$, so by the nth term test, the series diverges.

If $p > 0$ then $a_n = \frac{1}{n^p}$ is monotonically decreasing, so the Cauchy condensation theorem is applicable, that is, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{(2^k)^p}$ are both convergent or both divergent.

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{2^{-k}} \cdot \frac{1}{2^{k\,p}} = \sum_{k=1}^{\infty} \frac{1}{2^{(p-1)\,k}} = \sum_{k=1}^{\infty} \left(\left(\frac{1}{2}\right)^{p-1}\right)^k,$$

this is a geometric series with ratio $r = \left(\frac{1}{2}\right)^{p-1}$ and it is convergent if and only if

$$|\,r\,| = \left(\frac{1}{2}\right)^{p-1} < 1 \iff p - 1 > 0 \iff p > 1.$$