Calculus 1, 6th and 7th lecture

Orders of magnitudes

Definition: Suppose that $a_n \xrightarrow{n\to\infty} \infty$ and $b_n \xrightarrow{n\to\infty} \infty$. Then the order of magnitude of (a_n) is smaller than the order of magnitude of (b_n) if $\frac{a_n}{b_n} \xrightarrow{n\to\infty} 0$.

Notation: $a_n << b_n$.

Theorem: $n^n >> n! >> a^n >> n^k >> n^{\frac{1}{k}} >> \log n$, where $a > 1$ and $k \in \mathbb{N}^+$. That is,

a) $\lim\limits_{n\to\infty} \frac{n^n}{n!} = \infty$ **b)** $\lim\limits_{n\to\infty} \frac{n!}{a^n} = \infty$, where $a > 1$ **c)** $\lim\limits_{n\to\infty} \frac{a^n}{n} = \infty$, where $a > 1$

d) $\lim\limits_{n\to\infty} \frac{a^n}{n^k} = \infty$, where $a > 1$ and $k \in \mathbb{N}^+$ **e)** $\lim\limits_{n\to\infty} \frac{n}{\log_2 n} = \infty$

Some proofs. a) $\frac{n^n}{n!} = \frac{n}{1}\cdot\frac{n}{2}\cdot\frac{n}{3}\cdot\ldots\cdot\frac{n}{n-2}\cdot\frac{n}{n-1}\cdot\frac{n}{n} \geq n\cdot 1\cdot 1\cdot\ldots\cdot 1\cdot 1\cdot 1 = n \longrightarrow \infty \implies \frac{n^n}{n!} \longrightarrow \infty$

b) For example, if $a = 2$, then $\frac{n!}{2^n} = \frac{n}{2}\cdot\frac{n-1}{2}\cdot\frac{n-2}{2}\cdot\ldots\cdot\frac{3}{2}\cdot\frac{2}{2}\cdot\frac{1}{2} \geq \frac{n}{2}\cdot 1\cdot 1\cdot\ldots\cdot 1\cdot 1\cdot\frac{1}{2} = \frac{n}{4} \longrightarrow \infty \implies \frac{n!}{2^n} \longrightarrow \infty$

In general, if $a > 1$, then $\frac{n!}{a^n} = \frac{n}{a}\cdot\frac{n-1}{a}\cdot\ldots\cdot\frac{[a]+1}{a}\cdot\frac{[a]}{a}\cdot\ldots\cdot\frac{1}{a} \geq \frac{n}{a}\cdot 1\cdot\ldots\cdot 1\cdot c = \frac{c}{a}\cdot n \longrightarrow \infty \implies \frac{n!}{a^n} \longrightarrow \infty$,
where $c = \frac{[a]}{a}\cdot\ldots\cdot\frac{1}{a}$.

Remark: Binomial coefficients: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ where $k! = 1\cdot 2\cdot\ldots\cdot k$ and $0! = 1$.

Meaning: the number of subsets with k elements of a set with n elements.

Binomial theorem: $(a+b)^n = (a+b)\,(a+b)\ldots(a+b) = \sum\limits_{k=0}^{n}\binom{n}{k} a^k\, b^{n-k}$.

c) We will prove that $\lim\limits_{n\to\infty}\frac{a^n}{n} = \infty$, where $a = 1+\delta$ and $\delta > 0$. By the binomial theorem,

$(1+\delta)^n = \sum\limits_{k=0}^{n}\binom{n}{k}\delta^k = \binom{n}{0}\delta^0 + \binom{n}{1}\delta^1 + \binom{n}{2}\delta^2 + \ldots + \binom{n}{n}\delta^n \geq \binom{n}{2}\delta^2$, so

$\frac{(1+\delta)^n}{n} \geq \frac{\binom{n}{2}\delta^2}{n} = \frac{n(n-1)}{2n}\delta^2 = \frac{n-1}{2}\delta^2 \longrightarrow \infty \implies \frac{a^n}{n} \longrightarrow \infty$, where $a > 1$.

d) We will prove that $\lim\limits_{n\to\infty}\frac{a^n}{n^k} = \infty$, where $a > 1$ and $k \in \mathbb{N}^+$. This is a consequence of case c), since if $a > 1$ then $\sqrt[k]{a} > 1$ and $\frac{a^n}{n^k} = \left(\frac{\left(\sqrt[k]{a}\right)^n}{n}\right)^k$.

e) Let $a_n = \frac{n}{\log_2 n}$. It can be shown that (a_n) is monotonic increasing (we can prove this later) and

$$a_{2^k} = \frac{2^k}{\log_2 2^k} = \frac{2^k}{k} \longrightarrow \infty.$$

From these two properties it follows that $a_n \longrightarrow \infty$.

Example: $\dfrac{n^2 - 3^n}{n! + n^4} = \dfrac{3^n}{n!} \cdot \dfrac{\frac{n^2}{3^2} - 1}{1 + \frac{n^4}{n!}} \xrightarrow{n \to \infty} 0 \cdot \dfrac{0 - 1}{1 + 0} = 0.$

Theorem. $\lim\limits_{n \to \infty} n^k a^n = 0$, if $|a| < 1$ and $k \in \mathbb{N}^+$.

1st proof. It is a consequence of the following statements:

a) If $a_n \xrightarrow{n \to \infty} \infty$ then $\dfrac{1}{a_n} \xrightarrow{n \to \infty} 0$.

b) If $a > 1$ and $k \in \mathbb{N}^+$ then $\dfrac{a^n}{n^k} \xrightarrow{n \to \infty} \infty$.

c) If $\left| a_n \right| \xrightarrow{n \to \infty} 0$ then $a_n \xrightarrow{n \to \infty} 0$.

2nd proof. It is a consequence of the following statements:

(i) $\sqrt[n]{n} \xrightarrow{n \to \infty} 1$

(ii) If $0 < \lim\limits_{n \to \infty} \sqrt[n]{|a_n|} = L < 1$ then $a_n \xrightarrow{n \to \infty} 0$.

Proof of (ii): If $L \le q < 1$ then there exists $N \in \mathbb{N}$ such that for all $n > N$, $\sqrt[n]{|a_n|} < q$.

Then $0 < |a_n| < q^n \longrightarrow 0$ so by the Sandwich Theorem $a_n \xrightarrow{n \to \infty} 0$.

Using this, if $|a| < 1$ then $\sqrt[n]{|n^k a^n|} = \left(\sqrt[n]{n}\right)^k \cdot \left| a \right| \longrightarrow 1^k \cdot \left| a \right| < 1 \implies n^k a^n \longrightarrow 0$.

The Sandwich Theorem and two applications

Theorem (Sandwich Theorem). If $a_n \xrightarrow{n \to \infty} A \in \mathbb{R}$, $c_n \xrightarrow{n \to \infty} A \in \mathbb{R}$ and $a_n \le b_n \le c_n$ for all $n > N$, then $b_n \xrightarrow{n \to \infty} A \in \mathbb{R}$

Proof. Let $\varepsilon > 0$ be fixed. Then
there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $A - \varepsilon < a_n < A + \varepsilon$ and
there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $A - \varepsilon < c_n < A + \varepsilon$.
So if $n > \max\{N, N_1, N_2\}$ then
$A - \varepsilon < a_n \le b_n \le c_n < A + \varepsilon \implies |b_n - A| < \varepsilon$

Theorem. $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$.

1st proof. Apply the AM-GM inequality for $a_1 = \ldots a_{n-2} = 1$, $a_{n-1} = a_n = \sqrt{n}$.
Then

$$1 \le \sqrt[n]{n} = \sqrt[n]{1 \cdot \ldots \cdot 1 \cdot \sqrt{n} \cdot \sqrt{n}} \le \frac{(n-2) + 2\sqrt{n}}{n} \le 1 + \frac{2}{\sqrt{n}} \longrightarrow 1 + 0 - 0 = 1,$$

so by the Sandwich Theorem, $\sqrt[n]{n} \longrightarrow 1$.

2nd proof. Since $\sqrt[n]{n} \ge 1$ then we can write $\sqrt[n]{n} = 1 + \delta_n$, where $\delta_n \ge 0$. Then by the binomial theorem,

n can be estimated from below:

$$n = (1+\delta_n)^n = 1 + n\,\delta_n + \binom{n}{2}\delta_n^2 + \ldots + \binom{n}{n}\delta_n^2 \geq \binom{n}{2}\delta_n^2 = \frac{n(n-1)}{2}\,\delta_n^2$$

from where

$$0 \leq \delta_n \leq \sqrt{\frac{2}{n-1}} \longrightarrow 0,$$ so by the Sandwich Theorem, $\delta_n \longrightarrow 0$ and thus $\sqrt[n]{n} \longrightarrow 1$.

Theorem. If $p > 0$ then $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$.

1st proof. Assume that $p \geq 1$ and apply the AM-GM inequality for $a_1 = \ldots a_{n-2} = 1$, $a_{n-1} = a_n = \sqrt{p}$. Then

$$1 \leq \sqrt[n]{p} = \sqrt[n]{1 \cdot \ldots \cdot 1 \cdot \sqrt{p} \cdot \sqrt{p}} \leq \frac{(n-2) + 2\sqrt{p}}{n} \leq 1 + \frac{2\sqrt{p}}{n} \longrightarrow 1 + 0 = 1,$$

so by the Sandwich Theorem, $\sqrt[n]{p} \longrightarrow 1$.

If $0 < p < 1$, then $\frac{1}{p} > 1$, so $\sqrt[n]{p} = \frac{1}{\sqrt[n]{\frac{1}{p}}} \longrightarrow 1$.

2nd proof. If $p \geq 1$ then $\sqrt[n]{p} \geq 1$, so we can write $\sqrt[n]{p} = 1 + \delta_n$, where $\delta_n \geq 0$. Then by the binomial theorem, n can be estimated from below:

$$p = (1+\delta_n)^n = 1 + n\,\delta_n + \binom{n}{2}\delta_n^2 + \ldots + \binom{n}{n}\delta_n^2 \geq n\,\delta_n,$$

from where $0 \leq \delta_n \leq \frac{p}{n} \longrightarrow 0$, so by the Sandwich Theorem, $\delta_n \longrightarrow 0$ and thus $\sqrt[n]{p} \longrightarrow 1$.

The case $0 < p < 1$ is the same as before.

3rd proof. If $p \geq 1$ then $\sqrt[n]{p} \geq 1$, so we can write $\sqrt[n]{p} = 1 + \delta_n$, where $\delta_n \geq 0$. We show that $\delta_n \longrightarrow 0$. By the Bernoulli inequality

$$p = (1+\delta_n)^n \geq 1 + n\,\delta_n \implies \frac{p-1}{n} \geq \delta_n > 0$$

Since $\frac{p-1}{n} \longrightarrow 0$ then by the Sandwich Theorem $\delta_n \longrightarrow 0$, so $\sqrt[n]{p} \longrightarrow 1$.

The case $0 < p < 1$ is the same as before.

Monotonic sequences

Theorem. If (a_n) is monotonically increasing and not bounded above, then $a_n \overset{n\to\infty}{\longrightarrow} \infty$.

Proof. Let $P > 0$ be fixed. Since it is not an upper bound, there exists an $N \in \mathbb{N}$ such that $a_N > P$. By the monotonicity, if $n > N$ then $a_n \geq a_N > P$.

Consequence. If (a_n) is monotonically decreasing and not bounded below, then $a_n \overset{n\to\infty}{\longrightarrow} -\infty$.

Theorem. (1) If (a_n) is monotonically increasing and bounded above, then (a_n) is convergent and $\lim\limits_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

(2) If (a_n) is monotonically decreasing and bounded below, then (a_n) is convergent and $\lim\limits_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.

Proof of part (1). Let $A = \sup\{a_k : k \in \mathbb{N}\}$, then $a_n \leq A$ for all $n \in \mathbb{N}$.
Assume indirectly that $\lim\limits_{n\to\infty} a_n \neq A$. Then there exists $\varepsilon > 0$, such that for all $N \in \mathbb{N}$ there exists $n > N$, such that $a_n \leq A - \varepsilon$. By the monotonicity $a_N \leq a_n$, so $a_N \leq A - \varepsilon$ for all $N \in \mathbb{N}$. However, this is a contradiction, since A is the smallest upper bound of the sequence (so $A - \varepsilon$ is not an upper bound).
Therefore for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $A - \varepsilon < a_n \leq A < A + \varepsilon$, so $\lim\limits_{n\to\infty} a_n = A$.

The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$

Theorem. The sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$ is monotonically increasing and bounded, so it is convergent.

1st proof. a) Monotonicity. We use the inequality between the arithmetic and geometric means: if $a_1, a_2, \ldots, a_k \geq 0$ then $\sqrt[k]{a_1 a_2 \ldots a_k} \leq \dfrac{a_1 + a_2 + \ldots + a_k}{k}$.

Let $a_1 = \ldots = a_n = 1 + \dfrac{1}{n}$ and $a_{n+1} = 1$. Then

$$\sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n \cdot 1} \leq \frac{n\left(1 + \frac{1}{n}\right) + 1}{n+1} = 1 + \frac{1}{n+1},$$

so $a_n = \left(1 + \dfrac{1}{n}\right)^n \leq \left(1 + \dfrac{1}{n+1}\right)^{n+1} = a_{n+1}$ for all $n \in \mathbb{N}$.

b) Boundedness. We use the inequality between the arithmetic and geometric means for the numbers $a_1 = \ldots = a_n = 1 + \dfrac{1}{n}$ and $a_{n+1} = a_{n+2} = \dfrac{1}{2}$. Then

$$\sqrt[n+2]{\left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{4}} \leq \frac{n\left(1 + \frac{1}{n}\right) + 2 \cdot \frac{1}{2}}{n+2} = 1,$$

so $a_n = \left(1 + \dfrac{1}{n}\right)^n \leq 4$ for all $n \in \mathbb{N}$.

2nd proof with the binomial theorem (homework).

a) Boundedness. $a_n = \left(1 + \dfrac{1}{n}\right)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + 1 + \sum_{k=2}^{n} \frac{n(n-1)\ldots(n-(k-1))}{k!} \cdot \frac{1}{n^k} =$

$$= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdot \ldots \cdot \frac{n-(k-1)}{n} < 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot 1 \cdot \ldots \cdot 1 = \sum_{k=0}^{n} \frac{1}{k!} := s_n.$$

The sequence (s_n) is bounded above since the terms can be estimated from above by the terms of a geometric sequence with ratio $\dfrac{1}{2}$:

$$s_n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \ldots + \frac{1}{1\cdot 2\cdot \ldots\cdot n} <$$

$$< 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}\right) = 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 3 - \left(\frac{1}{2}\right)^{n-1} < 3.$$

So $a_n = \left(1 + \frac{1}{n}\right)^n < s_n = \sum_{k=0}^{n} \frac{1}{k!} < 3.$

b) Monotonicity.

$$a_{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{1}{n+1}\right)^k = 2 + \sum_{k=2}^{n+1} \frac{1}{k!} \cdot \frac{n+1}{n+1} \cdot \frac{n}{n+1} \cdot \frac{n-1}{n+1} \cdot \ldots \cdot \frac{(n+1)-(k-1)}{n+1} =$$

$$= 2 + \sum_{k=2}^{n+1} \frac{1}{k!} \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) \ldots \left(1 - \frac{k-1}{n+1}\right) =$$

$$= 2 + \sum_{k=2}^{n} \frac{1}{k!} \left(1 - \frac{1}{n+1}\right) \ldots \left(1 - \frac{k-1}{n+1}\right) + \binom{n+1}{n+1} \frac{1}{(n+1)^{n+1}} >$$

$$> 2 + \sum_{k=2}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \ldots \left(1 - \frac{k-1}{n}\right) + 0 = a_n$$

So $a_n < a_{n+1}$.

Definition: The sequence $\left(1 + \frac{1}{n}\right)^n$ is convergent so denote its limit by e:

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Remark: From the 2nd proof it follows that $2 < e < 3$.

Theorems:

1) e is irrational.

2) $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} = e$

3) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for all $x \in \mathbb{R}$

4) If $x_n \xrightarrow{n\to\infty} \infty$, then $\lim_{n\to\infty} \left(1 + \frac{1}{x_n}\right)^{x_n} = e$.

Examples for monotonic and bounded sequences

Example 1. Let $0 < a < 1$ and $b_n = a^n$, then $0 < b_{n+1} = a^{n+1} < a^n = b_n < 1$. Since (b_n) is bounded and monotonically decreasing then it is convergent, let $A = \lim_{n\to\infty} b_n$. Then

$$A = \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} a \cdot b_n = a \cdot A \iff A(1-a) = 0, \text{ so } A = 0.$$

Example 2. Let $a_1 = 4$ and $a_{n+1} = 8 - \frac{15}{a_n}$.

If (a_n) is convergent then $A = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = 8 - \frac{15}{A}$, so $A^2 - 8A + 15 = (A-3)(A-5) = 0$, therefore

$A = 3$ or $A = 5$.

If we prove that (a_n) is bounded and monotonically increasing or decreasing, then (a_n) is convergent and its limit is the supremum or the infimum of the sequence.

(i) First we prove boundedness by induction.

I. The statement is true for $n = 1$: $3 < a_1 = 4 < 5$.

II. Assume that $3 < a_n < 5$. Then

$$3 < a_n < 5 \Longrightarrow \frac{1}{5} < \frac{1}{a_n} < \frac{1}{3} \Longrightarrow 3 < \frac{15}{a_n} < 5 \Longrightarrow -3 > -\frac{15}{a_n} > -5$$

$$\Longrightarrow 3 < 8 - \frac{15}{a_n} = a_{n+1} < 5.$$

(ii) Next we prove monotonicity, also by induction.

I. $a_2 = \frac{17}{4} > a_1$

II. $a_n < a_{n+1} \Longrightarrow \frac{1}{a_n} > \frac{1}{a_{n+1}}$ (since $a_n > 0$) $\Longrightarrow -\frac{15}{a_n} < -\frac{15}{a_{n+1}}$

$$\Longrightarrow a_{n+1} = 8 - \frac{15}{a_n} < 8 - \frac{15}{a_{n+1}} = a_{n+2}.$$

Since (a_n) is monotonic increasing and bounded then a_n is convergent. The limit of (a_n) cannot be $A = 3$, since $a_1 = 4$ and the sequence is monotonic increasing. Therefore $\lim\limits_{n\to\infty} a_n = 5$.

Subsequences

Definition. Suppose $(n_k) : \mathbb{N} \longrightarrow \mathbb{N}$ is a strictly monotonically increasing sequence of natural numbers. Then we call the sequence (a_{n_k}) a **subsequence** of (a_n).

Examples: 1) The prime numbers are a subsequence of the positive integers.

2) $b_n = \frac{1}{1 + n^2}$ is a subsequence of $a_n = \frac{1}{1 + n}$ $(b_n = a_{n^2})$.

Remark. A subsequence can be obtained from a given sequence by deleting some or no elements without changing the order of the remaining elements.

Remark. If (n_k) is a strictly monotonically increasing sequence of natural numbers, then $n_k \overset{k\to\infty}{\longrightarrow} \infty$ since $n_k \geq n_1 + k - 1$.

Theorem. $\lim\limits_{n\to\infty} a_n = A$ if and only for all (a_{n_k}) subsequences $\lim\limits_{k\to\infty} a_{n_k} = A$.

Proof. 1) If all subsequences tend to the same limit A, then the subsequence $(a_{n+1}) \overset{n\to\infty}{\longrightarrow} A$ so for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_{n+1} - A| < \varepsilon$, so $|a_n - A| < \varepsilon$ if $n > N + 1$, so $a_n \overset{n\to\infty}{\longrightarrow} A$.
2) If (a_n) is convergent and (a_{n_k}) is a subsequence, then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that if $n > N$, then $|a_n - A| < \varepsilon$, and since $n_k \overset{k\to\infty}{\longrightarrow} \infty$, thus there exists $K \in \mathbb{N}$ such that if $k > K$, then $n_k > N$, so $|a_{n_k} - A| < \varepsilon$, therefore $a_{n_k} \overset{k\to\infty}{\longrightarrow} A$.