Calculus 1, 4th and 5th lecture

Sequences

Definition: A sequence is an $a : \mathbb{N} \longrightarrow \mathbb{R}$ mapping. Usual notation: $a(n) = a_n$ is the nth term of the sequence. The notation of the sequence is (a_n).

Definition: A sequence (a_n) is **monotonically increasing** if $\forall\, n \in \mathbb{N}\ (a_n \leq a_{n+1})$.
A sequence (a_n) is **monotonically decreasing** if $\forall\, n \in \mathbb{N}\ (a_n \geq a_{n+1})$.

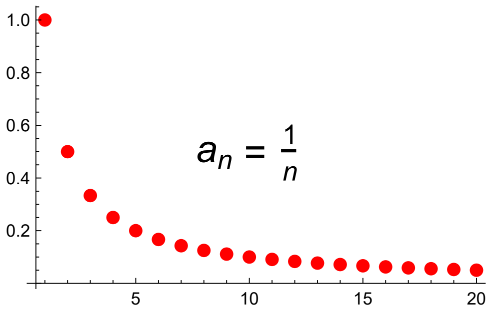
Definition: A sequence (a_n) is **bounded below (above)** if the set $\{a_n : n \in \mathbb{N}\}$ is bounded below (above).

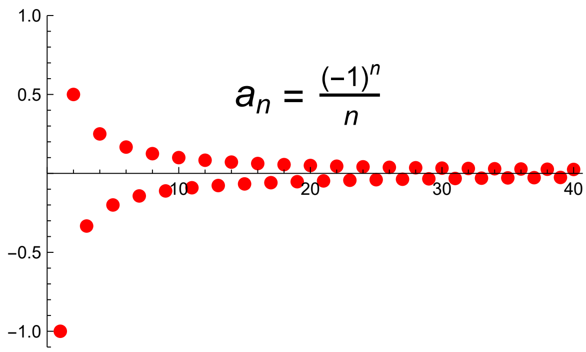
Convergent sequences

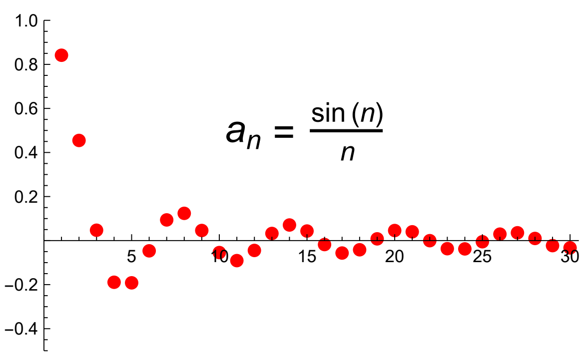
Definition: A sequence $(a_n) : \mathbb{N} \longrightarrow \mathbb{R}$ is **convergent**, and it tends to the limit $A \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists a threshold index $N(\varepsilon) \in \mathbb{N}$ such that for all $n > N(\varepsilon)$, $|\, a_n - A| < \varepsilon$.
Notation: $\lim\limits_{n \to \infty} a_n = A$ or $a_n \xrightarrow{n \to \infty} A$.
If a sequence if not convergent then it is **divergent**.

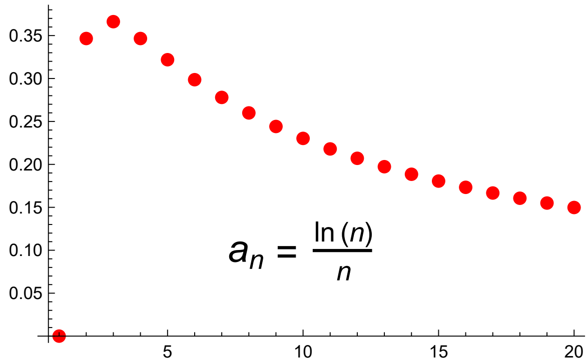
Remark: It is equivalent with the definition that for all $\varepsilon > 0$, the sequence has only finitely many terms outside of the interval $(A - \varepsilon,\ A + \varepsilon)$. (And the sequence has infinitely many terms in the interval.)

Examples for convergent sequences: 1) $a_n = \dfrac{1}{n}$ 2) $a_n = \dfrac{(-1)^n}{n}$ 3) $a_n = \dfrac{\sin(n)}{n}$ 4) $a_n = \dfrac{\ln(n)}{n}$

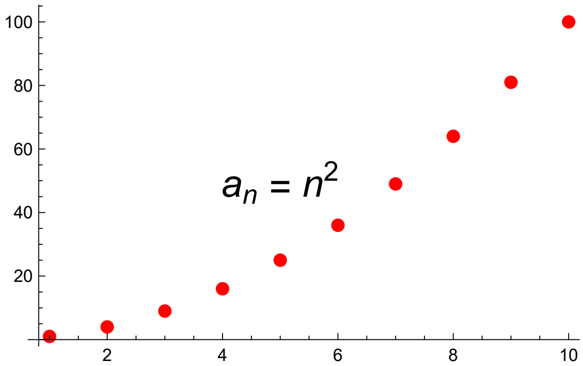


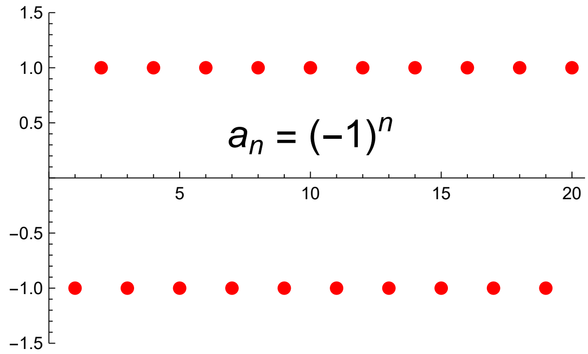


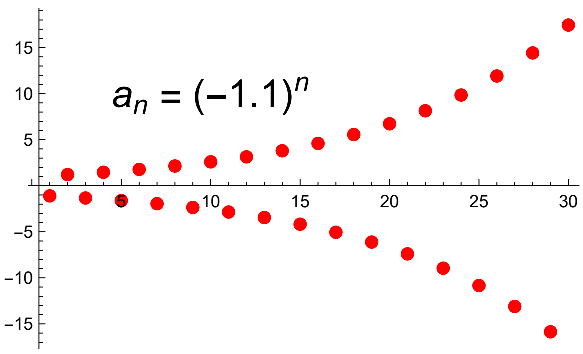


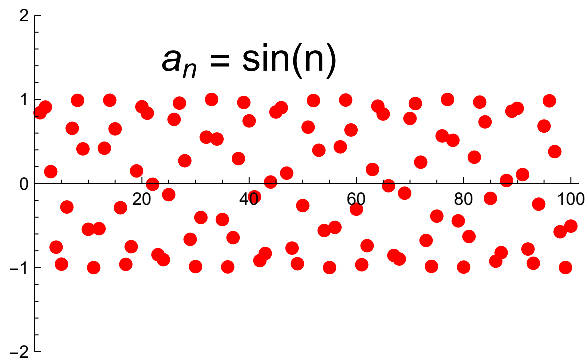


Examples for divergent sequences: 5) $a_n = n^2$ 6) $a_n = (-1)^n$ 7) $a_n = (-1.1)^n$ 8) $a_n = \sin(n)$









Exercises

1) Using the definition of the limit, show that a) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ b) $\lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Solution. Let $\varepsilon > 0$ be fixed. In both cases $|\,a_n - A\,| = \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$

so with the choice $N(\varepsilon) \geq \left[\frac{1}{\varepsilon}\right]$ the definition holds. For example, if $\varepsilon = 0.001$, then $N = 1000$ is a suitable threshold index.

2) Using the definition of the limit, show that $\lim\limits_{n\to\infty} \frac{6+n}{5.1-n} = -1$

Solution. Let $\varepsilon > 0$ be fixed. Then $\mid a_n - A \mid = \left| \frac{6+n}{5.1-n} - (-1) \right| = \left| \frac{11.1}{5.1-n} \right| \overset{\text{if } n>5}{=} \frac{11.1}{n-5.1} < \varepsilon \implies$
$n > 5.1 + \frac{11.1}{\varepsilon}$,
so $N(\varepsilon) \geq \left[5.1 + \frac{11.1}{\varepsilon}\right]$.

3) Using the definition of the limit, show that $\lim\limits_{n\to\infty} \frac{n^2-1}{2n^5+5n+8} = 0$

Solution. Let $\varepsilon > 0$ be fixed. Then $\mid a_n - A \mid = \left| \frac{n^2-1}{2n^5+5n+8} \right| = \frac{n^2-1}{2n^5+5n+8} < \varepsilon$.
This equation cannot be solved for n. However, it is not necessary to find the least possible threshold index, it is enough to show that a threshold index exists. So for the solution we use the transitive property of the inequalities, for example in the following
way:

$$\mid a_n - A \mid = \left| \frac{n^2-1}{2n^5+5n+8} \right| = \frac{n^2-1}{2n^5+5n+8} < \frac{n^2-0}{2n^5+0+0} < \frac{1}{2n^3} < \varepsilon \iff n > \sqrt[3]{\frac{1}{2\varepsilon}}, \text{ so } N(\varepsilon) \geq \left[\sqrt[3]{\frac{1}{2\varepsilon}}\right].$$

Here we estimated the fraction from above in such a way that we increased the numerator and decreased the denominator.

4) Using the definition of the limit, show that $\lim\limits_{n\to\infty} \frac{8n^4+3n+20}{2n^4-n^2+5} = 4$.

Solution. Let $\varepsilon > 0$ be fixed. Then $\mid a_n - A \mid = \left| \frac{8n^4+3n+20}{2n^4-n^2+5} - 4 \right| = \left| \frac{4n^2+3n}{2n^4-n^2+5} \right| =$

$$= \frac{4n^2+3n}{2n^4-n^2+5} < \frac{4n^2+3n^2}{2n^4-n^4+0} = \frac{7}{n^2} < \varepsilon \iff n > \sqrt{\frac{7}{\varepsilon}}, \text{ so } N(\varepsilon) \geq \left[\sqrt{\frac{7}{\varepsilon}}\right].$$

Divergent sequences

If a sequence if not convergent then it is **divergent**.
Example: Show that $a_n = (-1)^n$ is divergent.
Solution: Since the terms of the sequence are $-1, 1, -1, 1, \ldots$ then the possible limits are only 1 and -1. We show that $A = 1$ is not the limit.
For example for $\varepsilon = 1$, the interval $(A - \varepsilon, A + \varepsilon) = (0, 2)$ contains infinitely many terms (the terms a_{2n}), however, there are infinitely many terms outside of this interval (the terms a_{2n-1}). It means that there is no suitable threshold index $N(\varepsilon)$ for for $\varepsilon = 1$, so $A = 1$ is not the limit. Similarly, $A = -1$ is not the limit either, so the sequence is divergent.

Definition: The sequence $(a_n) : \mathbb{N} \longrightarrow \mathbb{R}$ tends to ∞ if for all $P > 0$ there exists a threshold index $N(P) \in \mathbb{N}$ such that for all $n > N(P)$, $a_n > P$.
Notation: $\lim\limits_{n\to\infty} a_n = \infty$ or $a_n \xrightarrow{n\to\infty} \infty$.

Definition: The sequence $(a_n) : \mathbb{N} \longrightarrow \mathbb{R}$ tends to $-\infty$ if for all $M < 0$ there exists a threshold index $N(M) \in \mathbb{N}$ such that for all $n > N(M)$, $a_n < M$.

Notation: $\lim\limits_{n\to\infty} a_n = -\infty$ or $a_n \xrightarrow{n\to\infty} -\infty$.

Remark: $\lim\limits_{n\to\infty} a_n = -\infty$ if and only if $\lim\limits_{n\to\infty} (-a_n) = \infty$.

Example: Let $a_n = 2n^3 + 3n + 5$. Show that $\lim\limits_{n\to\infty} a_n = \infty$.

Solution: Let $P > 0$ be fixed. Then $a_n = 2n^3 + 3n + 5 > 2n^3 > P \iff n > \sqrt[3]{\frac{P}{2}}$, so $N(P) \geq \left[\sqrt[3]{\frac{P}{2}}\right]$.

For example, if $P = 10^6$ then $N(P) = 80$ is a suitable threshold index.

Examples

Using the above definitions, the following statements can easily be proved:

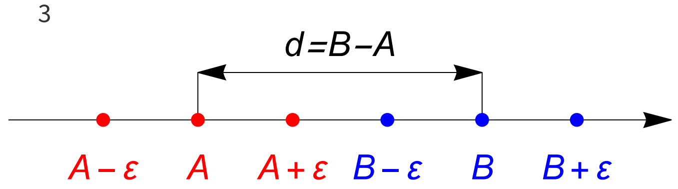
1) $\lim\limits_{n\to\infty} n^\alpha = \begin{cases} \infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$ **2)** $\lim\limits_{n\to\infty} a^n = \begin{cases} \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \\ \text{does not exist} & \text{if } a \leq -1 \end{cases}$

Theorems about the limit

Theorem (uniqueness of the limit): If $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} a_n = B$ then $A = B$.

Proof. We assume indirectly that $A \neq B$, for example $A < B$. Let $d = B - A$ and let

$\varepsilon = \frac{d}{3} > 0$.



Then because of the convergence of (a_n), there exist threshold indexes $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that
if $n > N_1$ then $A - \varepsilon < a_n < A + \varepsilon$ and
if $n > N_2$ then $B - \varepsilon < a_n < B + \varepsilon$.
But in this case if $n > \max\{N_1, N_2\}$ then $a_n < A + \varepsilon < B - \varepsilon < a_n$. This is a contradiction, so $A = B$.

Theorem: If (a_n) is convergent, then it is bounded.

Proof. Denote A the limit of (a_n). Then for $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that if $n > N$ then $A - \varepsilon < a_n < A + \varepsilon$. It means that the set $\{a_1, a_2, \ldots, a_N\}$ is finite, so the smallest element of $\{A - \varepsilon, a_1, \ldots, a_N\}$ is a lower bound and the largest element of $\{a_1, \ldots, a_N, A + \varepsilon\}$ of the set $\{a_n : n \in \mathbb{N}\}$.
Therefore for all n

$$\min\{A-\varepsilon, a_1, \ldots, a_N\} \le a_n \le \max\{a_1, \ldots, a_N, A+\varepsilon\}.$$

Remark. The converse of the statement is false, for example $a_n = (-1)^n$ is bounded but not convergent.

Operations with convergent sequences

Theorem 1. If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $b_n \xrightarrow{n\to\infty} B \in \mathbb{R}$ then $a_n + b_n \xrightarrow{n\to\infty} A + B$. (Sum Rule)

Proof. Let $\varepsilon > 0$ be fixed. Since $a_n \xrightarrow{n\to\infty} A$ and $b_n \xrightarrow{n\to\infty} B$, then for $\frac{\varepsilon}{2}$ there exists $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that if $n > N_1$, then $\left| a_n - A \right| < \frac{\varepsilon}{2}$ and if $n > N_2$, then $\left| b_n - B \right| < \frac{\varepsilon}{2}$. Thus, if $n > N = \max\{N_1, N_2\}$ then

$$\left| (a_n + b_n) - (A + B) \right| \le \left| a_n - A \right| + \left| b_n - B \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Here we used the triangle inequality: $|a + b| \le |a| + |b|$.

Theorem 2. If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $c \in \mathbb{R}$ then $c\, a_n \xrightarrow{n\to\infty} c\, A$. (Constant Multiple Rule)

Proof. Let $\varepsilon > 0$ be fixed.
(i) If $c = 0$ then the statement is trivial.
(ii) If $c \ne 0$ then because of the convergence of a_n, for $\frac{\varepsilon}{|c|}$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $\left| a_n - A \right| < \frac{\varepsilon}{|c|}$. Thus, if $n > N$ then

$$|c\, a_n - c\, A| = |c\,(a_n - A)| = \left| c \right| \cdot \left| a_n - A \right| < \left| c \right| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Here we used that $|a\, b| = |a|\, |b|$.

Consequence. (i) If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ then $-a_n \xrightarrow{n\to\infty} -A$. (Here $c = -1$.)
(ii) If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $b_n \xrightarrow{n\to\infty} B \in \mathbb{R}$ then $a_n - b_n = a_n + (-b_n) \xrightarrow{n\to\infty} A + (-B) = A - B$. (Difference Rule)

Theorem 3. (i) If $a_n \xrightarrow{n\to\infty} 0$ and $b_n \xrightarrow{n\to\infty} 0$ then $a_n\, b_n \xrightarrow{n\to\infty} 0$.
(ii) If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $b_n \xrightarrow{n\to\infty} B \in \mathbb{R}$ then $a_n\, b_n \xrightarrow{n\to\infty} A\, B$. (Product Rule)

Proof. Let $\varepsilon > 0$ be fixed.
(i) Since $a_n \xrightarrow{n\to\infty} 0$ and $b_n \xrightarrow{n\to\infty} 0$, then
for $\frac{\varepsilon}{2}$ there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $\left| a_n - 0 \right| < \frac{\varepsilon}{2}$ and
for 2 there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|b_n - 0| < 2$.
Thus, if $n > N = \max\{N_1, N_2\}$ then $|a_n\, b_n - 0| = \left| a_n \right| \cdot \left| b_n \right| < \frac{\varepsilon}{2} \cdot 2 = \varepsilon$.

(ii) It is obvious that if $c_n \equiv A$ for all $n \in \mathbb{N}$ (constant sequence) then $c_n \xrightarrow{n\to\infty} A$.

Thus $a_n - A \xrightarrow{n\to\infty} A - A = 0$ and $b_n - B \xrightarrow{n\to\infty} B - B = 0$.

Applying part (i) we get that $(a_n - A)(b_n - B) \xrightarrow{n\to\infty} 0$, that is, $a_n b_n - A b_n - B a_n + A B \xrightarrow{n\to\infty} 0$.

Then

$$a_n b_n = (a_n b_n - A b_n - B a_n + A B) + (A b_n + B a_n - A B) \xrightarrow{n\to\infty} 0 + (A B + A B - A B) = A B.$$

Theorem 4. If $a_n \xrightarrow{n\to\infty} 0$ and (b_n) is bounded then $a_n b_n \xrightarrow{n\to\infty} 0$.

Proof. Let $\varepsilon > 0$ be fixed.

Since (b_n) is bounded then there exists $K > 0$ such that $|b_n| < K$ for all $n \in \mathbb{N}$.

Since $a_n \xrightarrow{n\to\infty} 0$ then for $\frac{\varepsilon}{K}$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_n - 0| = \left| a_n \right| < \frac{\varepsilon}{K}$.

Thus, if $n > N$ then $|a_n b_n - 0| = \left| a_n \right| \cdot \left| b_n \right| < \frac{\varepsilon}{K} \cdot K = \varepsilon$.

Theorem 5. If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ then $\left| a_n \right| \xrightarrow{n\to\infty} |A|$.

Proof. $\big|\, |a_n| - |A| \,\big| \le |a_n - A| < \varepsilon$ if $n > N(\varepsilon)$.

Remark. The converse of the statement is not true. For example, $a_n = (-1)^n$ is divergent but $|a_n| = 1^n = 1 \xrightarrow{n\to\infty} 1$.

However, the following statement is true: $\left| a_n \right| \xrightarrow{n\to\infty} 0 \implies a_n \xrightarrow{n\to\infty} 0$.

Since $\big|\, |a_n| - 0 \,\big| = |a_n| = |a_n - 0| < \varepsilon$ if $n > N(\varepsilon)$.

Theorem 6. (i) If $b_n \xrightarrow{n\to\infty} B \neq 0$ then $\frac{1}{b_n} \xrightarrow{n\to\infty} \frac{1}{B}$.

(ii) If $a_n \xrightarrow{n\to\infty} A \in \mathbb{R}$ and $b_n \xrightarrow{n\to\infty} B \neq 0$ then $\frac{a_n}{b_n} \xrightarrow{n\to\infty} \frac{A}{B}$. (Quotient Rule)

Proof. (i) First, by the convergence of (b_n) and by theorem 5, $\left| b_n \right| \xrightarrow{n\to\infty} |B| \neq 0$ and thus there exists $N_1 = N_1\left(\frac{|B|}{2}\right) \in \mathbb{N}$ such that if $n > N_1$ then $\left| \left| b_n \right| - \left| B \right| \right| < \frac{|B|}{2} \iff$ $\left| B \right| - \frac{|B|}{2} < \left| b_n \right| < \left| B \right| + \frac{|B|}{2}$. Then $\left| b_n \right| > \frac{|B|}{2}$ for all $n > N_1$.

Second, for a fixed $\varepsilon > 0$ there exists $N_2 = N_2\left(\frac{|B|^2 \varepsilon}{2}\right) \in \mathbb{N}$ such that if $n > N_2$ then $\left| b_n - B \right| < \frac{|B|^2 \varepsilon}{2}$.

Therefore, if $n > N = \max\{N_1, N_2\}$ then

$$\left| \frac{1}{b_n} - \frac{1}{B} \right| = \left| \frac{B - b_n}{B \cdot b_n} \right| = \frac{|B - b_n|}{|B| \cdot |b_n|} < \frac{1}{\left| B \right| \cdot \frac{|B|}{2}} \cdot \frac{|B|^2 \varepsilon}{2} = \varepsilon.$$

(ii) By theorem 3 and theorem 6, part (i): $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \xrightarrow{n\to\infty} A \cdot \frac{1}{B} = \frac{A}{B}$

Remark. By induction it can be proved that Theorem 1 and Theorem 3 can be generalized to the sum and product of **finitely many** convergent sequences. However, they are not true for infinitely many terms, as the following examples show.

Examples. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{10} = 1^{10} = 1$ or $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{k} = 1^k = 1$, where $k \in \mathbb{N}^+$ is a fixed constant, independent of n. However, $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n} \neq 1^n = 1$. Later we will see that $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n} = e$.

Example. $a_n = \dfrac{1}{n^2} + \dfrac{2}{n^2} + \ldots + \dfrac{500}{n^2} \longrightarrow 0 + 0 + \ldots + 0 = 0$

The number of the terms is 500 which is independent of n and thus applying Theorem 1 finitely many times, the correct result is 0.

Example. $b_n = \dfrac{1}{n^2} + \dfrac{2}{n^2} + \ldots + \dfrac{n}{n^2} \longrightarrow 0 + 0 + \ldots + 0 = 0$ is a WRONG SOLUTION!

Since $b_1 = \dfrac{1}{1^2}$, $b_2 = \dfrac{1}{2^2} + \dfrac{2}{2^2}$, $b_3 = \dfrac{1}{3^2} + \dfrac{2}{3^2} + \dfrac{3}{3^2}$, $b_4 = \dfrac{1}{4^2} + \dfrac{2}{4^2} + \dfrac{3}{4^2} + \dfrac{4}{4^2}$, ...,

it can be seen that the number of the terms depends on n, so b_n is not the sum of finitely many sequences and thus Theorem 1 cannot be generalized to this case. The correct solution is:

$$b_n = \frac{1+2+\ldots+n}{n^2} = \frac{(1+n)\cdot\frac{n}{2}}{n^2} = \frac{1+n}{2n} = \frac{\frac{1}{n}+1}{2} \longrightarrow \frac{0+1}{2} = \frac{1}{2}$$

Theorem 7. If $a_n \geq 0$ and $a_n \xrightarrow{n\to\infty} A \geq 0$ then $\sqrt{a_n} \xrightarrow{n\to\infty} \sqrt{A}$.

Proof. Let $\varepsilon > 0$ be fixed.

(i) If $a_n \xrightarrow{n\to\infty} A = 0$ then there exists $N_1 = N_1(\varepsilon^2) \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - 0| = a_n < \varepsilon^2$.

Therefore, if $n > N_1$ then $\left|\sqrt{a_n} - 0\right| = \sqrt{a_n} < \varepsilon$.

(ii) If $a_n \xrightarrow{n\to\infty} A > 0$ then there exists $N_2 = N_2\left(\varepsilon\sqrt{A}\right) \in \mathbb{N}$ such that if $n > N_2$ then $\left|a_n - A\right| < \varepsilon\sqrt{A}$.

Therefore, if $n > N_2$ then

$$\left|\sqrt{a_n} - \sqrt{A}\right| = \left|\frac{a_n - A}{\sqrt{a_n} + \sqrt{A}}\right| = \frac{|a_n - A|}{\sqrt{a_n} + \sqrt{A}} \leq \frac{|a_n - A|}{0 + \sqrt{A}} < \frac{\varepsilon\sqrt{A}}{\sqrt{A}} = \varepsilon.$$

Remark. If $a_n \xrightarrow{n\to\infty} A \geq 0$ then $\sqrt[k]{a_n} \xrightarrow{n\to\infty} \sqrt[k]{A}$ for all $k \in \mathbb{N}^+$.

It can be proved by using the following identity: $a^k - b^k = (a-b)\left(a^{k-1} + a^{k-2}b + \ldots + a\,b^{k-1} + b^{k-1}\right)$.

Example. $\dfrac{2^{2n} + \cos(n^2)}{4^{n+1} - 5} = \dfrac{4^n}{4^n} \cdot \dfrac{1 + \left(\frac{1}{4}\right)^n \cdot \cos(n^2)}{4 - 5\cdot\left(\frac{1}{4}\right)^n} \longrightarrow \dfrac{1+0}{4-0} = \dfrac{1}{4}$.

Additional theorems about the limit

Theorem. If $a_n \xrightarrow{n\to\infty} \infty$ then $\dfrac{1}{a_n} \xrightarrow{n\to\infty} 0$.

Proof. Let $\varepsilon > 0$ be fixed. Since $a_n \xrightarrow{n\to\infty} \infty$, then for $P = \frac{1}{\varepsilon}$ there exists $N \in \mathbb{N}$ such that if $n > N$ then

$a_n > \frac{1}{\varepsilon}$,

so $\left| \frac{1}{a_n} - 0 \right| = \frac{1}{a_n} < \varepsilon$.

Question: Is it true that if $a_n \xrightarrow{n\to\infty} 0$ then $\frac{1}{a_n} \xrightarrow{n\to\infty} \infty$?

Answer: No, for example, if $a_n = -\frac{2}{n} \longrightarrow 0$ then $\frac{1}{a_n} = -\frac{n}{2} \longrightarrow -\infty$.

Or, if $a_n = \left(-\frac{1}{2}\right)^n \longrightarrow 0$ then for $b_n = \frac{1}{a_n} = (-2)^n$, $b_{2k} \longrightarrow \infty$ and $b_{2k} \longrightarrow -\infty$, so $\lim\limits_{n\to\infty} \frac{1}{a_n} \neq \infty$.

However, the following statements hold.

Theorem. a) If $a_n > 0$ and $a_n \xrightarrow{n\to\infty} 0$ then $\frac{1}{a_n} \xrightarrow{n\to\infty} \infty$. Notation: $\frac{1}{0+} \longrightarrow +\infty$.

b) If $a_n < 0$ and $a_n \xrightarrow{n\to\infty} 0$ then $\frac{1}{a_n} \xrightarrow{n\to\infty} -\infty$. Notation: $\frac{1}{0-} \longrightarrow -\infty$.

c) If $a_n \xrightarrow{n\to\infty} 0$ then $\frac{1}{|a_n|} \xrightarrow{n\to\infty} \infty$.

Proof. a) Let $P > 0$ be fixed. Since $0 < a_n \xrightarrow{n\to\infty} 0$, then for $\varepsilon = \frac{1}{P}$ there exists $N \in \mathbb{N}$ such that if $n > N$ then

$a_n = |a_n - 0| < \frac{1}{P}$, so $\frac{1}{a_n} > P$.

b), c): homework.

Theorem. If $a_n \xrightarrow{n\to\infty} \infty$ and $b_n \geq a_n$ for $n > N$, then $b_n \longrightarrow \infty$.

Proof. Let $P > 0$ be fixed. Since $a_n \xrightarrow{n\to\infty} \infty$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $a_n > P$. So if $n > \max\{N, N_1\}$ then $b_n > P$.

Consequence. Suppose that $a_n \xrightarrow{n\to\infty} \infty$, $b_n \xrightarrow{n\to\infty} \infty$, $c_n \xrightarrow{n\to\infty} c > 0$ and $|d_n| \leq K$ for all $n > \in \mathbb{N}$. Then

a) $a_n + b_n \xrightarrow{n\to\infty} \infty$

b) $a_n \cdot b_n \xrightarrow{n\to\infty} \infty$

c) $c_n \cdot a_n \xrightarrow{n\to\infty} \infty$

d) $a_n + d_n \xrightarrow{n\to\infty} \infty$

Proof. a) Since $a_n \xrightarrow{n\to\infty} \infty$, it may be assumed that there exists $N \in \mathbb{N}$ such that $a_n \geq 0$ for $n > N$.

Then $a_n + b_n \geq b_n \xrightarrow{n\to\infty} \infty$, so $a_n + b_n \xrightarrow{n\to\infty} \infty$.

b) Since $a_n \xrightarrow{n\to\infty} \infty$ and $b_n \xrightarrow{n\to\infty} \infty$, it may be assumed that there exists $N \in \mathbb{N}$ such that $a_n \geq 1$ and $b_n \geq 0$ for $n > N$.

Then $a_n \cdot b_n \geq b_n \xrightarrow{n\to\infty} \infty$, so $a_n \cdot b_n \xrightarrow{n\to\infty} \infty$.

c) Let $P > 0$ be fixed.

Since $c_n \xrightarrow{n\to\infty} c > 0$ then there exists $N_1 = N_1\left(\frac{c}{2}\right) N \in \mathbb{N}$ such that $c_n > \frac{c}{2}$ if $n > N_1$.

Since $a_n \xrightarrow{n\to\infty} \infty$ then there exists $N_2 = N_2\left(\frac{2P}{c}\right) N \in \mathbb{N}$ such that $a_n > \frac{2P}{c}$ if $n > N_2$.

So if $n > \max\{N_1, N_2\}$ then $c_n \cdot a_n > \frac{2P}{c} \cdot \frac{c}{2} = P$.

d) Let $P > 0$ be fixed. $a_n + d_n \geq a_n - K > P$ if and only if $a_n > K + P$.

Since $a_n \xrightarrow{n\to\infty} \infty$ then for $K + P$ there exists $N \in \mathbb{N}$ such that $a_n > K + P$ if $n > N$.

Then for $n > N$, $a_n + d_n > P$ also holds, so $a_n + d_n \xrightarrow{n\to\infty} \infty$.

Example. $5n^2 + 2^n \cdot n - (-1)^n \xrightarrow{n\to\infty} \infty$.

Remark. The above statements can be denoted in the following way:

a) $\infty + \infty \longrightarrow \infty$ **b)** $\infty \cdot \infty \longrightarrow \infty$ **c)** $c \cdot \infty \longrightarrow \infty$ (where $c > 0$) **d)** ∞ + bounded $\longrightarrow \infty$.

Similar statements can be proved, for example, $\frac{0}{\infty} \longrightarrow 0$, $\frac{\text{bounded}}{\infty} \to 0$, $\frac{\infty}{+0} \to \infty$, $\frac{\infty}{-0} \to -\infty$.

The meaning of $\frac{0}{\infty} \longrightarrow 0$ is that if $a_n \xrightarrow{n\to\infty} 0$ and $b_n \xrightarrow{n\to\infty} \infty$ then $\frac{a_n}{b_n} \longrightarrow 0$.

Such statements are summarized in the following tables where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ denotes the extended set of real numbers.

Out[∘]=

Addition:

lim(a_n)	lim(b_n)	lim(a_n+b_n)
$a \in \mathbb{R}$	$b \in \mathbb{R}$	$a + b$
∞	$b \in \mathbb{R}$	∞
$-\infty$	$b \in \mathbb{R}$	$-\infty$
∞	∞	∞
$-\infty$	$-\infty$	$-\infty$
∞	$-\infty$	?

Subtraction:

lim(a_n)	lim(b_n)	lim(a_n−b_n)
$a \in \mathbb{R}$	$b \in \mathbb{R}$	$a - b$
∞	$b \in \mathbb{R}$	∞
$-\infty$	$b \in \mathbb{R}$	$-\infty$
∞	$-\infty$	∞
∞	∞	?
$-\infty$	$-\infty$	?

Out[∘]=

Multiplication

lim(a_n)	lim(b_n)	lim($a_n b_n$)
$a \in \mathbb{R}$	$b \in \mathbb{R}$	ab
∞	$b > 0$	∞
∞	$b < 0$	$-\infty$
$-\infty$	$b > 0$	$-\infty$
$-\infty$	$b < 0$	∞
∞	∞	∞
∞	$-\infty$	$-\infty$
$-\infty$	$-\infty$	∞
∞	0	?
$-\infty$	0	?

Division:

lim(a_n)	lim(b_n)	lim($\frac{a_n}{b_n}$)
$a \in \mathbb{R}$	$b \in \mathbb{R}\setminus\{0\}$	$\frac{a}{b}$
∞	$b > 0$	∞
∞	$b < 0$	$-\infty$
$-\infty$	$b > 0$	$-\infty$
$-\infty$	$b < 0$	∞
$a \in \mathbb{R}$	$\pm\infty$	0
0	$b \in \overline{\mathbb{R}}$, $b \neq 0$	0
$a \in \overline{\mathbb{R}}$, $a \neq 0$	0	$\lvert . \rvert = \infty$
0	0	?
$\pm\infty$	$\pm\infty$	?

The meaning of $\lvert . \rvert$ is that $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \infty$.

Undefined forms: $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, 1^∞, ∞^0, 0^0

Examples:

1) Limit of the form $\infty - \infty$:

$a_n = n^2,\ b_n = n,\ a_n - b_n = n^2 - n \to \infty$

$a_n = n,\ b_n = n,\ a_n - b_n = n - n = 0 \to 0$

$a_n = n,\ b_n = n^2,\ a_n - b_n = n - n^2 \to -\infty$

2) Limit of the form $0 \cdot \infty$:

$\frac{1}{n} \cdot n^2 = n \to \infty,\ \frac{1}{n} \cdot n = 1 \to 1,\ \frac{1}{n^2} \cdot n = \frac{1}{n} \to 0,\ \frac{(-1)^n}{n} \cdot n = (-1)^n$ (it doesn't have a limit)

3) Limit of the form $\frac{\infty}{\infty}$:

$\frac{n}{n^2} = \frac{1}{n} \to 0,\ \frac{n^2}{n} = n \to \infty,\ \frac{n^2}{n^2} = 1 \to 1$

4) Limit of the form $\frac{0}{0}$:

$\frac{\frac{1}{n}}{\frac{1}{n^2}} = n \to \infty,\ \frac{\frac{1}{n^2}}{\frac{1}{n}} = \frac{1}{n} \to 0,\ \frac{\frac{1}{n}}{\frac{1}{n}} = 1 \to 1,\ \frac{(-1)^n \frac{1}{n}}{\frac{1}{n^2}} = (-1)^n \cdot n$ (it doesn't have a limit)

Wolframalpha

Some examples:

$$\lim_{n \to \infty} \frac{n^2 + 5n}{3n^2 - 7}$$

https://www.wolframalpha.com/input/?i=limit+%28n%5E2%2B5n%29%2F%283n%5E2-7%29+as+n-%3Einfinity

$$\lim_{n \to \infty} \left(\sqrt{n^2 - 3n} - \sqrt{n^2 + 1} \right)$$

https://www.wolframalpha.com/input/?i=limit+sqrt%28n%5E2-3n%29-sqrt%28n%5E2%2B1%29+as+n-%3Einfinity

$$\lim_{n \to \infty} \left(\sqrt[3n]{2n} \right)$$

https://www.wolframalpha.com/input/?i=limit+%282n%29%5E%281%2F%283n%29%29+as+n-%3Einfinity

$$\lim_{n \to \infty} \frac{\binom{n}{2}}{\binom{n}{3}}$$

https://www.wolframalpha.com/input/?i=limit+%28n+choose+2%29%2F%28n+choose+3%29+as+n-%3Einfinity

$$\lim_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \ldots + \frac{1}{n}}$$

https://www.wolframalpha.com/input/?i=limit+%28sum+1%2Fk%2C+k%3D1+to+n%29%5E%281%2Fn%29+as+n-%3Einfinity