## Calculus 1, 3rd lecture

## Axioms for the real numbers

$\mathbb{R}$ is a set whose elements are called real numbers.
Two operations, called addition and multiplication are defined in $\mathbb{R}$ such that $\mathbb{R}$ is closed under these operations, that is, $\forall a, b \in \mathbb{R}(a+b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R})$.

## Addition:

1) $\forall a, b \in \mathbb{R}(a+b=b+a)$ (commutativity),
2) $\forall a, b, c \in \mathbb{R}((a+b)+c)=a+(b+c))$ (associativity),
3) $\exists 0 \in \mathbb{R}(\forall a \in \mathbb{R}(a+0=0+a=0))$ (existence of zero element),
4) $\forall a \in \mathbb{R}(\exists b \in \mathbb{R}(a+b=0))$ (existence of additive inverse, notation: $b=-a)$.

## Multiplication:

5) $\forall a, b \in \mathbb{R}(a \cdot b=b \cdot a) \quad$ (commutativity),
6) $\forall a, b, c \in \mathbb{R}((a \cdot b) \cdot c=a \cdot(b \cdot c))$ (associativity),
7) $\exists 1 \in \mathbb{R}(\forall a \in \mathbb{R}(a \cdot 1=1 \cdot a=a)) \quad$ (existence of unit element),
8) $\forall a \in \mathbb{R}(\exists b \in \mathbb{R} \backslash\{0\}(a \cdot b=1))$ (existence of multiplicative inverse, notation: $b=a^{-1}$ ).

For the two operations above:
9) $\forall a, b, c \in \mathbb{R}(a+b) \cdot c=a \cdot c+b \cdot c \quad$ (the multiplication is distributive with respect to the addition).

Axioms (1)-(9) are the axioms for a field.

## Ordering:

10) Exactly one of the following is true: $a<b, b<a, a=b$ (trichotomy),
11) $\forall a, b, c \in \mathbb{R}((a<b) \wedge(b<c)) \Longrightarrow(a<c) \quad$ (transitivity),
12) $\forall a, b, c \in \mathbb{R}((a<b) \wedge c>0) \Longrightarrow a \cdot c<b \cdot c$
13) $\forall a, b, c \in \mathbb{R}(a<b) \Longrightarrow a+c<b+c$ (monotonicity)

Axioms (1)-(13) are the axioms for an ordered field.

## Archimedian axiom:

14) $\forall a \in \mathbb{R}(\exists n \in \mathbb{N}(a<n))$

So far we can substitute $\mathbb{R}$ with $\mathbb{Q}$. But there is also:

## Cantor axiom:

15) $a_{1}, b_{1}, a_{2}, b_{2}, \ldots \in \mathbb{R}$
$\left(\forall n \in \mathbb{N}\left(a_{n} \leq a_{n+1} \wedge b_{n+1} \leq b_{n}\right)\right) \Longrightarrow\left(\exists x \in \mathbb{R}\left(\forall n \in \mathbb{N}\left(x \in\left[a_{n}, b_{n}\right]\right)\right)\right)$
$\left(\right.$ so $\left.\bigcap_{n=1}^{\infty}\left[a_{n}, b_{n}\right] \neq \varnothing\right)$.

It states that any nested sequence of closed intervals has a non-empty intersection.

Example: Let $a_{1}=1.4<a_{2}=1.41<a_{3}=1.414<a_{4}=1.4142<\ldots$ and $b_{1}=1.5>b_{2}=1.42>b_{3}=1.415>b_{4}=1.4143>\ldots$
$\left(a_{n}=\left[10^{n} \cdot \sqrt{2}\right] \cdot 10^{-n}, b_{n}=\left(\left[10^{n} \cdot \sqrt{2}\right]+1\right) \cdot 10^{-n}\right.$, where [.] denotes the floor function.)
Then $\bigcap_{n=1}^{\infty}\left[a_{n}, b_{n}\right]=\{\sqrt{2}\} \in \mathbb{R} \backslash \mathbb{Q}$.
Remark. Closeness is important, for example if $I_{n}=\left(0, \frac{1}{n}\right]$ then $\bigcap_{n=1}^{\infty} I_{n}=\varnothing$.

## Consequences

Some elementary laws of algebra and inequalities follow from the axioms.
For example:

1) For all $a \in \mathbb{R}$, exactly one of the following properties hold: $a>0, a=0, a<0$.
( $a>0 \Longleftrightarrow-a<0$ )
2) $(a<b) \wedge(c<d) \Longrightarrow a+c<b+d$

Specifically: $(a>0) \wedge(b>0) \Longrightarrow a+b>0$
3) $(0 \leq a<b) \wedge(0 \leq c<d) \Longrightarrow a c<b d$

Specifically: $(a>0) \wedge(b>0) \Longrightarrow a b>0$
4) $(a<b) \wedge(c<0) \Longrightarrow a c>b c$

Specifically: $a<b \Longrightarrow-a>-b$
5) (i) $0<a<b \Rightarrow \frac{1}{a}>\frac{1}{b}$
(ii) $a<b<0 \Rightarrow \frac{1}{a}>\frac{1}{b}$
(iii) $a<0<b \Longrightarrow \frac{1}{a}<\frac{1}{b}$
(i) and (ii): $(a<b) \wedge(a b>0) \Longrightarrow \frac{1}{a}>\frac{1}{b}$
(iii): $(a<b) \wedge(a b<0) \Longrightarrow \frac{1}{a}<\frac{1}{b}$
6) For all $a, b \in \mathbb{R},|a+b| \leq|a|+|b|$ and $||a|-|b|| \leq|a-b|$.
7) If $n$ is a positive integer and $0<a<b$ then $a^{n}<b^{n}$.
8) $\forall x \in \mathbb{R} \quad(x \cdot 0=0)$
9) $\forall x \in \mathbb{R} \quad(x \cdot y=0 \Longrightarrow x=0$ or $y=0)$

Proof of 8):
$x \cdot 0=x \cdot 0+0=x \cdot 0+(x \cdot 0-x \cdot 0)=(x \cdot 0+x \cdot 0)-x \cdot 0=x \cdot(0+0)-x \cdot 0=x \cdot 0-x \cdot 0=0$.

Proof of 9):
$x \neq 0 \Longrightarrow y=1 \cdot y=((1 / x) \cdot x) \cdot y=(1 / x) \cdot(x \cdot y)=(1 / x) \cdot 0=0$.

## Bounded subsets of real numbers

Definition. $A \subset \mathbb{R}$ is bounded above if there exists a $K \in \mathbb{R}$ such that $a \leq K$ if $a \in A$.
$(\exists K \in \mathbb{R}(\forall a \in A(a \leq K)))$.) In this case $K$ is an upper bound of $A$.

Definition. $A \subset \mathbb{R}$ is bounded below if there exists a $k \in \mathbb{R}$ such that $a \geq k$ if $a \in A$. $(\exists k \in \mathbb{R}(\forall a \in A(a \geq k)))$.) In this case $k$ is a lower bound of $A$.

Definition. $A \subset \mathbb{R}$ is bounded if it is has an upper bound and a lower bound. It means that there exists a $K>0$ such that $|a|<K$ for all $a \in A$.

Examples: 1) $\mathbb{N}$ is bounded below
2) $(0,1]=\{x \in \mathbb{R}: 0<x \leq 1\}$ is bounded (for example, upper bounds are
$1,3, \pi, \ldots$, lower bounds are $0,-3,-100, \ldots$ )
3) $\mathbb{Q}$ has no upper bound or lower bound

Remark: A bounded set has infinitely many lower and upper bounds.

Definition. If a set $A$ is bounded above, then the supremum of $A$ is the least upper bound of $A$ (Notation: sup $A$ ). If $A$ is not bounded above, then $\sup A=\infty$.

Definition. If a set $A$ is bounded below then the infimum of $A$ is the greatest lower bound of $A$ (Notation: $\inf A$ ). If $A$ is not bounded below, then $\inf A=-\infty$.

Examples: 1$) \inf \mathbb{N}=1, \sup \mathbb{N}=\infty$
2) $\inf (0,1]=0, \sup (0,1]=1$
3) $\inf \mathbb{Q}=-\infty, \sup \mathbb{Q}=\infty$

Definition. The minimum of the set $A$ is $l$ if $l \in A$ and $l=\inf A$.
The maximum of the set $A$ is $h$ if $h \in A$ and $h=\sup A$.

Examples: 1) The minimum of $\mathbb{N}$ is 1 and it has no maximum.
2) The maximum of $(0,1]$ is 1 and it has no minimum.
3) $\mathbb{Q}$ has no minimum and no maximum.

## Least-upper-bound property

Theorem (Least-upper-bound property, Dedekind):
If a non-empty subset of $\mathbb{R}$ is bounded above then it has a least upper bound in $\mathbb{R}$.

Consequence. If a non-empty subset of $\mathbb{R}$ is bounded below then it has a greatest lower bound in $\mathbb{R}$.

Remarks. 1) In the above system of axioms, the axioms of Cantor and Archimedes can be replaced by this statement.
2) The set of rational numbers does not have the least-upper-bound property under the usual order. For example, $\left\{x \in \mathbb{Q}: x^{2} \leq 2\right\}=\mathbb{Q} \cap(-\sqrt{2}, \sqrt{2})$ has an upper bound in $\mathbb{Q}$ but does not have a least upper bound in $\mathbb{Q}$ since $\sqrt{2}$ is irrational.

## Complex numbers

Definition. The complex field $\mathbb{C}$ is the set of ordered pairs of real numbers: $\mathbb{C}=\mathbb{R}^{2}=\{(a, b): a, b \in \mathbb{R}\}$ with addition and multiplication defined by

$$
\begin{aligned}
& (a, b)+(c, d)=(a+c, b+d) \\
& (a, b)(c, d)=(a c-b d, a d+b c)
\end{aligned}
$$

Remark. Commutativity and associativity of addition and multiplication as well as distributivity (see 1), 2), 5), 6), 9) ) follow easily from the same properties of reals numbers.
3 ) the additive identity or zero element is $(0,0)$
4) the additive inverse of $(a, b)$ is $(-a,-b)$
7) the multiplicative identity or unit element is $(1,0)$
8) the multiplicative inverse of $(a, b) \neq(0,0)$ can be found in the following way:
$\begin{aligned} &(a, b)(x, y)=(1,0) \Longleftrightarrow a x-b y=1 \Longleftrightarrow x=\frac{a}{a^{2}+b^{2}}, y=\frac{-b}{a^{2}+b^{2}} \\ & b x+a y=0\end{aligned}$

Thus the complex numbers form a field.
Remark. We associate the complex number of the form $(a, 0)$ with the corresponding real number $a$. Then

$$
\begin{aligned}
& \left(a_{1}, 0\right)+\left(a_{2}, 0\right)=\left(a_{1}+a_{2}, 0\right) \text { corresponds to } a_{1}+a_{2} \text { and } \\
& \left(a_{1}, 0\right)\left(a_{2}, 0\right)=\left(a_{1} a_{2}, 0\right) \text { corresponds to } a_{1} a_{2} .
\end{aligned}
$$

Since

$$
(0,1)(0,1)=(-1,0)=-1
$$

we can say that $(0,1)$ is a square root of -1 and it will be denoted by $i$.

The algebraic form of complex numbers. We can rewrite any complex number in the following way:

$$
(a, b)=(a, 0)+(0, b)=a+b i
$$

where $a, b \in \mathbb{R}$ and $i^{2}=-1$.

Addition: $(a+b i)+(c+d i)=(a+b)+(c+d) i$
Multiplication: $(a+b i)(c+d i)=a c+b d i^{2}+a d i+b c i=(a c-b d)+(a d+b c) i$

## The complex plane.

To each complex number $z=a+b i$ we associate the point $(a, b)$ in the Cartesian plane. Real numbers are thus associated with points on the $x$-axis, called the real axis and the purely imaginary numbers bi correspond to points on the $y$-axis, called the imaginary
axis.


Definitions. If $z=a+b i$ then

- the real part of $z$ is $\quad \operatorname{Re}(z)=a \in \mathbb{R}$
- the imaginary part of $z$ is $\operatorname{Im}(z)=b \in \mathbb{R}$
- the conjugate of $z$ is $\quad \bar{z}=a-b i$
- the absolute value or modulus of $z$ is $|z|=\sqrt{a^{2}+b^{2}} \geq 0$ (the length of the vector $z$ ) - the argument of $z$, defined for $z \neq 0$, is the angle which the vector originating from 0 to $z$ makes with the positive $x$-axis. Thus $\arg (z)=\varphi(\operatorname{modulo} 2 \pi)$ for which

$$
\cos \varphi=\frac{\operatorname{Re}(z)}{|z|} \text { and } \cos \varphi=\frac{\operatorname{Im}(z)}{|z|}
$$

Some identities:

$$
\begin{aligned}
& \operatorname{Re}(z)=\frac{z+\bar{z}}{2}, \quad \operatorname{Im}(z)=\frac{z-\bar{z}}{2 i}, \quad \overline{\bar{z}}=z, \\
& z \bar{z}=(a+b i)(a-b i)=a^{2}-b^{2} i^{2}=a^{2}+b^{2}=|z|^{2}
\end{aligned}
$$

$$
\overline{z_{1} \pm z_{2}}=\overline{z_{1}} \pm \overline{z_{1}}, \quad \overline{z_{1} \cdot z_{2}}=\overline{z_{1}} \cdot \overline{z_{1}}, \quad \overline{\left(\frac{z_{1}}{z_{2}}\right)}=\frac{\overline{z_{1}}}{\overline{z_{2}}}
$$

## The trigonometric form (or polar form) of complex numbers.

Let $z=a+b i \neq 0, r=|z|$ and $\varphi=\arg (z)$. Then $a=r \cos \varphi$ and $b=r \sin \varphi$ and

$$
z=r(\cos \varphi+i \sin \varphi)
$$

where $r$ and $\varphi$ are called the polar coordinates of $z$.

Multiplication and division: Let $z_{1}=r_{1}\left(\cos \varphi_{1}+i \sin \varphi_{1}\right)$ and $z_{2}=r_{2}\left(\cos \varphi_{2}+i \sin \varphi_{2}\right)$. Then

$$
\begin{aligned}
& z_{1} z_{2}=r_{1} r_{2}\left(\cos \left(\varphi_{1}+\varphi_{2}\right)+i \sin \left(\varphi_{1}+\varphi_{2}\right)\right) \\
& \frac{z_{1}}{z_{2}}=\frac{r_{1}}{r_{2}}\left(\cos \left(\varphi_{1}-\varphi_{2}\right)+i \sin \left(\varphi_{1}-\varphi_{2}\right)\right) \quad\left(\text { if } r_{2} \neq 0\right)
\end{aligned}
$$

Reciprocal, conjugation, $n$th power: Let $z=r(\cos \varphi+i \sin \varphi)$. Then

$$
\begin{aligned}
& \frac{1}{z}=\frac{1}{r}(\cos (-\varphi)+i \sin (-\varphi)) \quad(\text { if } r \neq 0) \\
& \bar{z}=r(\cos (-\varphi)+i \sin (-\varphi)) \\
& z^{n}=r^{n}(\cos (n \varphi)+i \sin (n \varphi)) \quad\left(n \in \mathbb{N}^{+}\right)
\end{aligned}
$$

If $r \neq 0$ then it holds for $n \in \mathbb{Z}$.
$n$th root: If $z \neq 0$ and $n \in \mathbb{N}^{+}$then $w \in \mathbb{C}$ is an $n$th root of $z$ if $w^{n}=z$. Then
$w=\sqrt[n]{r(\cos \varphi+i \sin \varphi)}=\sqrt[n]{r}\left(\cos \frac{\varphi+k \cdot 2 \pi}{n}+i \sin \frac{\varphi+k \cdot 2 \pi}{n}\right)$ where $k=0,1, \ldots, n-1$.

Some identities:
$\left|z_{1} z_{2}\right|=\left|z_{1}\right| \cdot\left|z_{2}\right|,\left|\frac{1}{z}\right|=\frac{1}{|z|}, \quad\left|\frac{z_{1}}{z_{2}}\right|=\frac{\left|z_{1}\right|}{\left|z_{2}\right|}, \quad\left|z^{n}\right|=|z|^{n}, \quad|\bar{z}|=|z|$

Fundamental theorem of algebra: Every degree $n$ polynomial with complex coefficients has exactly $n$ complex roots, if counted with multiplicity.

## Exercise

1. Using the field and ordering axioms prove that $\forall a \in \mathbb{R} a^{2} \geq 0$.
2. Show that no ordering can make the field of complex numbers into an ordered field.

Solution: See exercises 1.1.8 and 1.1.9 here:
http://etananyag.ttk.elte.hu/FiLeS/downloads/4b_FeherKosToth_MathAnExII.pdf

## Wolframalpha

Some examples:

1) $\sqrt[4]{-16}$
https://www.wolframalpha.com/input/?i=\(-16\)\^\(1\%2F4\)
2) $\frac{\sqrt{i}}{1-i}$
https://www.wolframalpha.com/input/?i=sqrt\(i\)\%2F\(1-i\)
3) $z^{2}=\bar{z}$
https://www.wolframalpha.com/input/?i=z\^2\%3Dconjugate\(z\)
4) $\operatorname{Re}\left(z^{2}\right)=2 \operatorname{Im}(z), \operatorname{Im}\left(z^{2}\right)=2 \operatorname{Re}(z)$
https://www.wolframalpha.com/input/?i=Re\(z\^2\)\%3D2Im\(z\)\%2C+Im\(z\^2\% 29\%3D2Re\%28z\%29
