Calculus 1, 1st and 2nd lecture

Logic

https://www.whitman.edu/mathematics/higher_math_online/section01.01.html

Basic ideas of logic: proposition and logical values
A **proposition** is a statement that is either true or false but not both.
The **truth value** of a proposition is either **true** (denoted as T) or **false** (denoted as F).
When we deal with propositions in logic we consider sentences without being interested in their meanings, only examining them as true or false statements. In the following, propositions will be denoted by P, Q etc.

Example: If $P(x, y)$ is "$x^2 + y = 12$", then $P(2, 8)$ and $P(3, 3)$ are true, while $P(1, 4)$ and $P(0, 6)$ are false.

Propositions can be combined together using **logical connectives** or **logical operations**.

Most common logical connectives:

1) Negation (logical NOT): $\neg P$
Read "not P", "the denial of P". $\neg P$ is true if and only P is false.

Truth table:

P	¬ P
T	F
F	T

2) Conjunction (logical AND): $P \wedge Q$
Read "P and Q".
$P \wedge Q$ is true if and only if both P and Q are true.

3) Disjunction (logical OR): $P \vee Q$
Read "P or Q".
$P \vee Q$ is true if and only if at least one of P and Q are true. (This is an inclusive OR, that is, “either or both”.)

4) Implication: $P \Longrightarrow Q$ (or $P \longrightarrow Q$).
Read "if P then Q", "P implies Q".
$P \Longrightarrow Q$ is true if and only if both P and Q are true or if P is false and Q is arbitrary.
P: hypothesis, premise, antecedent
Q: conclusion, consequence

Other terminologies:
"P is sufficient for Q", "Q is necessary for P", "Q only if P", "Q when P", "Q follows from P" etc.

5) Biconditional (logical equivalence): $P \Longleftrightarrow Q$ (or $P \longleftrightarrow Q$)
Read "P if and only if Q", "P is necessary and sufficient for Q".

Abbreviation: "P iff Q".
$P \Longleftrightarrow Q$ is true if and only if both P and Q have the same truth value.

Truth tables:

P	Q	P ∨ Q	P ∧ Q	P ⟹ Q	P ⟺ Q
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	T	F
F	F	F	F	T	T

A few important logical identities:
1) $P \equiv \neg(\neg P)$
2) $P \vee Q \equiv Q \vee P$
3) $P \wedge Q \equiv Q \wedge P$ (2,3: commutativity)
4) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
5) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ (4,5: associativity)
6) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
7) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ (6,7: distributivity)
8) $(P \Longrightarrow Q) \equiv (\neg P \vee Q)$
9) $(P \Longrightarrow Q) \equiv (\neg Q \Longrightarrow \neg P)$
10) $(P \Longleftrightarrow Q) \equiv (P \Longrightarrow Q) \wedge (Q \Longrightarrow P)$

Proof: by truth tables. For example, the proof of 8):

P	Q	$\neg P$	$\neg P \vee Q$	$P \Longrightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan's laws:

$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$
$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$

We can use De Morgan's laws to simplify the denial of $P \Longrightarrow Q$:

$$\neg(P \Longrightarrow Q) \equiv \neg(\neg P \vee Q) \equiv (\neg\neg P) \wedge (\neg Q) \equiv P \wedge \neg Q$$

The denial of $P \Longleftrightarrow Q$:

$$\neg(P \Longleftrightarrow Q) \equiv \neg((P \Longrightarrow Q) \wedge (Q \Longrightarrow P)) \equiv (\neg(P \Longrightarrow Q)) \vee (\neg(Q \Longrightarrow P)) \equiv$$
$$\equiv (P \wedge (\neg Q)) \vee (Q \wedge (\neg P)) \equiv (P \vee Q) \wedge ((\neg P) \vee (\neg Q))$$

A proposition which is always true is called **tautology**, for example: $P \vee (\neg P)$ or $(P \Longrightarrow Q) \Longleftrightarrow (\neg Q \Longrightarrow \neg P)$.
A proposition which is always false is called **contradiction**, for example: $P \wedge (\neg P)$.

Quantifiers

The phrase "for every x" (sometimes "for all x") is called a **universal quantifier** and is denoted by $\forall x$.
The phrase "there exists an x such that" is called an **existential quantifier** and is denoted by $\exists x$.

The universal quantifier

A sentence $\forall x\ P(x)$ is true if and only if $P(x)$ is true no matter what value is substituted for x.
Example: $\forall x \forall y\ (x + y = y + x)$, i.e., the commutative law of addition.

The universal quantifier is frequently encountered in the following context: $\forall x\ (P(x) \Longrightarrow Q(x))$, which may be read, "All x satisfying $P(x)$ also satisfy $Q(x)$".
Parentheses are crucial here; there is a difference between $\forall x\, P(x) \Longrightarrow \forall x\, Q(x)$ and $(\forall x\, P(x)) \Longrightarrow Q(x)$.

Examples:

1) "All squares are rectangles" symbolically should be written as: $\forall x$ (x is a square $\Longrightarrow$ x is a rectangle).

This construction sometimes is used to express a mathematical sentence of the form **"if this, then that"**:
2) If we say, "if x is negative, so is its cube", we usually mean "every negative x has a negative cube".
Symbolically: $\forall x\big((x < 0) \Longrightarrow (x^3 < 0)\big)$.
3) "If two numbers have the same square, then they have the same absolute value" should be written as
$\forall x \forall y\big((x^2 = y^2) \Longrightarrow (\,|\,x\,| = |\,y\,|)\big)$.

The existential quantifier

A sentence $\exists x\, P(x)$ is true if and only if there is at least one value of x that makes $P(x)$ true.
Example:
$\exists x\big(x \geq x^2\big)$ is true since $x = 0$ is a solution. There are many others.

The existential quantifier is frequently encountered in the following context: $\exists x\ (P(x) \wedge Q(x))$, which may be read, "Some x satisfying $P(x)$ also satisfies $Q(x)$."
Example: $\exists x$(x is a prime number $\wedge$ x is even) , i.e., "some prime number is even."
Note: This is not the same as $\exists x(P(x) \Rightarrow Q(x))$. To see why this does not work, suppose $P(x)$ ="x is an apple" and $Q(x)$ ="x is an orange." The sentence "some apples are oranges" is certainly false, but $\exists x\ (P(x) \Rightarrow Q(x))$ is true. To see this suppose x_0 is some particular orange. Then $P(x_0) \Rightarrow Q(x_0)$ evaluates to $F \Rightarrow T$, which is T, and the existential quantifier is satisfied.

The statement "no x satisfying $P(x)$ satisfies $Q(x)$" can be written $\forall x(P(x) \Rightarrow \neg Q(x))$.
(This is equivalent to $\neg \exists x(P(x) \wedge Q(x))$.)
Example: "No triangles are rectangles," can be written $\forall x$ (x is a triangle $\Rightarrow$ x is not a rectangle).

Negations of propositions

Statement:	**Negation:**
P or Q	not P and not Q

P and Q	not P or not Q
if P, then Q	P and not Q
For all x, $P(x)$	There exists x such that not $P(x)$
There exists x such that $P(x)$	For every x, not $P(x)$

When quantifiers occur in the proposition, as a general rule, we have to change ∀ into ∃ and change ∃ into ∀, and finally negate the proposition which follows the quantifier.

Exercise

Negate the following statements:
a) Every bear loves honey.
b) There is a type of honey that not all bears love.
c) On every floor there is at least one window that is open.
d) In every building there is at least one floor where all the windows are open.
e) Every sailor knows a port where there is a pub he hasn't been to before.

Sets

A set is a collection of objects; any one of the objects in a set is called a **member** or an **element** of the set. If x is an element of a set A we write $x \in A$.

Two sets are **equal** if and only if they have the same elements.
Example: $\{1, 2\} = \{2, 1\} = \{1, 1, 2\} = \{1, 2, 2, 1, 2, 1\}$

The **empty set** is the set without elements: $\emptyset = \{\}$.
Note that $\emptyset \neq \{\emptyset\}$: : the first contains nothing, the second contains a single element, namely the empty set.

Definition of sets: $\{x \in \text{universal set} \mid \text{conditions for } x\}$
$\{x \in \text{universal set} : \text{conditions for } x\}$

Example: $\{x \in \mathbb{Z} : x > 0\}$: the set of positive integers
$\{x \in \mathbb{Z} : \exists\, n \in \mathbb{Z}\ (x = 2\,n)\}$: the set of even integers

Notations

Real numbers: $\mathbb{R}$
Positive real numbers: $\mathbb{R}^+$
Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ (non-negative integers) or
$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ (positive integers).
Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Positive integers: $\mathbb{N}^+$ (or $\mathbb{Z}^+$)
Rational numbers (fractions): $\mathbb{Q} = \left\{x \in \mathbb{R} \;\middle|\; \exists\, k \in \mathbb{Z} \quad \exists\, n \in \mathbb{Z} \setminus \{0\} \quad \left(x = \frac{k}{n}\right)\right\}$

Intervals: for example $[a; b] = \{x \in \mathbb{R} \mid a \le x \le b\}$, $[a; b[= \{x \in \mathbb{R} \mid a \le x < b\}$ etc.;
$]a; +\infty[= \{x \in \mathbb{R} \mid a < x\}$, $]-\infty; b] = \{x \in \mathbb{R} \mid x \le b\}$ etc.
Instead of $]a; b[$, $]a; b]$ etc. $(a; b)$, $(a; b]$ is also used.

A is a **subset** of B if $\forall x(x \in A \Rightarrow x \in B)$. Notation: $A \subseteq B$.
The sets A and B are **equal** if and only if $A \subseteq B$ and $A \subseteq B$, that is, $\forall x(x \in A \Leftrightarrow x \in B)$.
A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$. (There exists at least one element $a \in A$ such that $a \notin B$.)
Notation: $A \subset B$. Example: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Operations with sets

The **union** of A and B: $A \cup B = \{x \mid x \in A \vee x \in B\}$
The **intersection** of A and B: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
The **difference** of A and B: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$
The **complement** of A contains all the elements in the universal set that are not included in A:
$A^c = \overline{A} = \{x \in U \mid x \notin A\} = U \setminus A$.
A and B are **disjoint** if $A \cap B = \emptyset$.

Example: Suppose $U = \{1, 2, 3, \ldots, 10\}$, $A = \{1, 3, 4, 5, 7\}$, $B = \{1, 2, 4, 7, 8, 9\}$; then
$A^c = \overline{A} = \{2, 6, 8, 9, 10\}$, $A \cap B = \{1, 4, 7\}$ and $A \cup B = \{1, 2, 3, 4, 5, 7, 8, 9\}$.

Cartesian product

If $a, b \in U$ we can form the **ordered pair** (a, b).
Definition: $(a, b) = \{\{a\}, \{a, b\}\}$.
The fundamental property of ordered pairs is that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. The order matters: $(1, 2) \neq (2, 1)$.
The **Cartesian product** of the sets A and B: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Example: $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is the plane, $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$ is the 3-dimensional space.

Some identities

1) Commutativity: $A \cap B = B \cap A$, $A \cup B = B \cup A$
2) Associativity: $(A \cap B) \cap C = A \cap (B \cap C)$, $(A \cup B) \cup C = A \cup (B \cup C)$
3) Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4) De Morgan's laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$, $\overline{A \cup B} = \overline{A} \cap \overline{B}$

5) $A \cap B \subseteq A$
6) $A \subseteq A \cup B$
7) $A \subseteq B \iff \overline{B} \subseteq \overline{A}$

Families of sets

https://www.whitman.edu/mathematics/higher_math_online/section01.06.html

Proofs

https://www.whitman.edu/mathematics/higher_math_online/chapter02.html

Direct proofs

Since $((P \Longrightarrow R) \wedge (R \Longrightarrow Q)) \Longrightarrow (P \Longrightarrow Q)$ is always true (it is a tautology), we can prove $P \Longrightarrow Q$ by proving $P \Longrightarrow R$ and $R \Longrightarrow Q$ where R is any other proposition.

Example: Inequality of arithmetic and geometric means:

If $a, b \geq 0$ then $\sqrt{a\,b} \leq \frac{a+b}{2}$ and equality holds if and only if $a = b$.

Proof: $\frac{a+b}{2} \geq \sqrt{a\,b} \iff (a+b)^2 \geq 4\,a\,b \iff a^2 - 2\,a\,b + b^2 \geq 0 \iff (a-b)^2 \geq 0$, which always holds.

Indirect proof

There are two methods of indirect proof: proof of the contrapositive and proof by contradiction. They are closely related, even interchangeable in some circumstances. What unites them is that they both start by assuming the denial of the conclusion.

Proof of the contrapositive

We can prove $P \Longrightarrow Q$ by proving its **contrapositive**, $\neg Q \Longrightarrow \neg P$.
These are logically equivalent: $(P \Longrightarrow Q) \equiv ((\neg P) \vee Q) \equiv ((\neg Q) \Longrightarrow (\neg P))$.
In the proof we assume that Q is false and try to prove that P is false.
For example, the contrapositive of “If it is Sunday, I go to church” is “If I am not going to church, it is not Sunday.”

Example: If $a\,b$ is even then either a or b is even: Assume both a and b are odd. Since the product of odd numbers is odd, $a\,b$ is odd.

Proof by contradiction

To prove a sentence P by contradiction we assume $\neg P$ and derive a statement that is known to be false. This means P must be true.
If we want to prove a statement of the form $P \Longrightarrow Q$ then we assume that P is true and Q is false (since $\neg (P \Longrightarrow Q) \equiv \neg (\neg P \vee Q) \equiv P \wedge \neg Q$) and try to derive a statement known to be false. Note that this statement need not be $\neg P$, this is the principal difference between proof by contradiction and proof of the contrapositive.
Remark: If R is any false proposition, then $P \Longrightarrow Q \equiv (P \wedge (\neg Q)) \Longrightarrow R$, thus we can prove $P \Longrightarrow Q$ by proving $(P \wedge (\neg Q)) \Longrightarrow R$.

Examples. In the following two examples we will use the **fundamental theorem of arithmetic** also known as **unique factorization theorem** which states that every integer greater than 1 can be factored uniquely as a product of primes, up to the order of factors.

1) Theorem: There are infinitely many primes.

Proof. Assume there are only finitely many primes $p_1, p_2, \ldots, p_k$ and let $n = p_1 p_2 \ldots p_k + 1$.
Then n is not divisible by any of the primes $p_1, p_2, \ldots, p_k$ since the remainder is always 1. It means that n is either another prime or it has a prime factor different from $p_1, p_2, \ldots, p_k$.
This is a contradiction since we started from the fact that there are exactly k primes and then came to the conclusion that there must be at least one more prime.
It means that there are infinitely many primes.

2) $\sqrt{3}$ is irrational.

Proof. Assume indirectly that $\sqrt{3}$ is rational. Then is can be written in the form $\sqrt{3} = \frac{a}{b}$ where a, b are integers and $b \neq 0$. From this we get that $3\,b^2 = a^2$. Consider the exponent of 3 in the prime factorization of both sides. Since in the prime factorization of a square number all exponents are even, it means that the exponent of 3 is odd on the left-hand side and even on the right-hand side. However, this contradicts the unique factorization theorem, so $\sqrt{3}$ is irrational.

Induction

Let $P(n)$ denote a statement that depends on the natural number n.
A proof by induction consists of two cases.
1) The **base case** (or basis) proves that $P(n_0)$ is true without assuming any knowledge of other cases.
2) The **induction step** proves that if $P(k)$ is true for any natural number k then $P(k+1)$ must also be true. These two steps establish that $P(n)$ holds for all natural numbers $n \geq n_0$.

Example: Prove by induction that for every positive integer n the following statement holds:
$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

Solution: 1) Base case: the statement is true for $n = 1$ since $1 = \frac{1 \cdot 2}{2}$.
2) Induction step: Assume that the statement holds for $n = k$, that is,
$1 + 2 + \ldots + k = \frac{k(k+1)}{2}$ (this is the induction hypothesis).
Using this, we prove that the statement holds for $n = k + 1$, that is,
$1 + 2 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$:

$$1 + 2 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.$$

It means that the statement holds for all positive integer n, that is, $n_0 = 1$.

Inequalities

Triangle inequality

$$|a+b| \le |a| + |b|$$

Proof. Since both sides are negative, squaring is an equivalent transformation:
$|a+b| \le |a| + |b| \iff a^2 + 2ab + b^2 \le a^2 + 2|ab| + b^2 \iff 2ab \le 2|ab|$
This is always true since $x \le |x|$ for all $x \in \mathbb{R}$.

Inequalitiy of arithmetic and geometric means

If $a_1, a_2, \ldots a_n \ge 0$ then $\sqrt[n]{a_1 a_2 \ldots a_n} \le \frac{a_1 + a_2 + \ldots + a_n}{n}$ and equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Proof: by induction.

a) The statement holds for $n = 2$ (see direct proof above): $\frac{a_1 + a_2}{2} \ge \sqrt{a_1 a_2}$.

b) We prove that if the statement is true for n then it is also true for $2n$. For this, divide the arbitrarily fixed $2n$ numbers into two groups of n. Apply the induction hypothesis for these two groups and then apply part a) for case $n = 2$.

$$\frac{a_1 + \ldots + a_{2n}}{2n} = \frac{1}{2}\left(\frac{a_1 + \ldots + a_n}{n} + \frac{a_{n+1} + \ldots + a_{2n}}{n}\right) \ge \frac{1}{2}\left(\sqrt[n]{a_1 \ldots a_n} + \sqrt[n]{a_{n+1} \ldots a_{2n}}\right) \ge \sqrt[2n]{a_1 \ldots a_{2n}}.$$

Thus, the statement holds for $n = 2^k$.

c) Using a kind of reverse induction, we prove that if the statement holds for $(n + 1)$ then it is also true for n and thus it holds for all positive integers.

Let $a_{n+1} = \frac{a_1 + \ldots + a_n}{n} = A_n$ and apply the statement for the $(n + 1)$ numbers $a_1, \ldots, a_n, a_{n+1}$. With equivalent steps, we get

$$A_n = \frac{a_1 + \ldots + a_n + A_n}{n+1} \ge \sqrt[n+1]{a_1 \ldots a_n A_n} \iff A_n^{n+1} \ge a_1 \ldots a_n A_n \iff A_n^n \ge a_1 \ldots a_n$$

$$\iff A_n \ge \sqrt[n]{a_1 \ldots a_n}.$$

d) Finally, we prove the equality part of the theorem.

If $a_1 = \ldots a_n = a$ then the equality obviously holds since $\frac{a_1 + \ldots + a_n}{n} = a = \sqrt[n]{a_1 \ldots a_n}$.

Now suppose that for example $a_1 \ne a_2$. Using that in this case $\frac{a_1 + a_2}{2} > \sqrt{a_1 a_2}$, we get

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} = \frac{\frac{a_1 + a_2}{2} + \frac{a_1 + a_2}{2} + a_3 + \ldots + a_n}{n} \ge$$

$$\ge \sqrt[n]{\left(\frac{a_1 + a_2}{2}\right)^2 a_3 \ldots a_n} > \sqrt[n]{\left(\sqrt{a_1 a_2}\right)^2 a_3 \ldots a_n} = \sqrt[n]{a_1 \ldots a_n}.$$

HM-GM-AM-QM inequalities

The inequalities between the harmonic mean, geometric mean, arithmetic mean and quadratic mean of the positive real numbers $a_1, a_2, \ldots, a_n$:

$$0 < \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n} \leq \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}}$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Bernoulli inequality

If $x \geq -1$ then $(1 + x)^n \geq 1 + n\,x$.

Proof: By induction.
1) For $n = 1$: $1 + x \leq 1 + x$.
2) Assume that $(1 + x)^n \geq 1 + n\,x$ and multiply both sides by $1 + x \geq 0$:
$(1 + x)^{n+1} \geq (1 + n\,x) \cdot (1 + x) = 1 + (n + 1)\,x + n\,x^2 \geq 1 + (n + 1)\,x$.

Exercises

Logic

1. Prove the following identities.
a) $p \Leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
b) $\neg (p \Leftrightarrow q) \equiv (p \vee q) \wedge (\neg p \vee \neg q)$
c) $(p \wedge q \wedge r) \Rightarrow s \equiv p \Rightarrow [q \Rightarrow (r \Rightarrow s)]$
d) $[\neg p \wedge (p \Rightarrow q)] \Rightarrow \neg p \equiv T$
e) $(p \wedge q) \Rightarrow p \equiv p \Rightarrow (p \vee q)$
f) $\neg [((p \wedge q) \vee r) \Rightarrow p] \Leftrightarrow q \equiv (p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee \neg q)$

3. There are goats and fleas in a yard. $\Phi(B, K)$ denotes that a flea has bitten a goat. Write down the denial of the following statements with formula and text.

a) $(\forall K)(\exists B)\,\Phi(B, K)$ *b*) $(\exists K)(\forall B)\,\Phi(B, K)$
c) $(\exists B)(\forall K)\,\Phi(B, K)$ *d*) $(\forall K)(\forall B)\,\Phi(B, K)$
e) $(\exists K)(\exists B)\,\Phi(B, K)$ *f*) $(\forall B)(\exists K)\,\Phi(B, K)$

4. Are the following sentences true or false? Write down the negation of the statements.
a) $\forall x \forall y (x < y \implies x^2 < y^2)$ b) $\forall x \exists y\,(y > x)$
c) $x > 0 \implies \exists y\,(x = y^2)$ d) $\forall x \forall y \exists z\,(x = y^z)$
e) $[x \in \mathbb{N} \wedge y \in (\mathbb{N} \setminus \{1, x\})] \implies x/y \notin \mathbb{N}$ f) $\exists x < 0 \exists y < 0\,(x^2 + x\,y + y^2 = 3)$

Sets

1. For the given universe U and the given sets A and B, find $\overline{A}$, $A \cap B$ and $A \cup B$.
a) $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 5, 8\}$, $B = \{2, 3, 5, 6\}$

b) $U = \mathbb{R},\ A = (-\infty, 2],\ B = (-1, \infty)$
c) $U = \mathbb{Z},\ A = \{n : n \text{ is even}\},\ B = \{n : n \text{ is odd}\}$
d) $U = \mathbb{Q},\ A = \emptyset,\ B = \{q : q > 0\}$
e) $U = \mathbb{N},\ A = N,\ B = \{n : n \text{ is even}\}$
f) $U = \mathbb{R},\ A = (-\infty, 0],\ B = [-2, 3)$
g) $U = \mathbb{N},\ A = \{n : n \leq 6\},\ B = \{1, 2, 4, 5, 7, 8\}$
h) $U = \mathbb{R} \times \mathbb{R},\ A = \{(x, y) : x2 + y2 \leq 1\},\ B = \{(x, y) : x \geq 0,\ y \geq 0\}$

2. Let A, B, C arbitrary sets. Prove the following identities.
a) $(A \setminus B) \cup B = A \cup B$
b) $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$
c) $A = (A \cup B) \cap (A \cup B)$
d) $(A \setminus B) \cup (B \setminus C) \cup (C \setminus A) \cup (A \cap B \cap C) = A \cup B \cup C$
e) If $A \subset C$ then $A \setminus B = A \cap (C \setminus B)$.
f) $(A \setminus B) \cup B = A$ if and only if $B \subset A$.
g) $(A \setminus B) \cup C = (A \cup C) \setminus (B \cup C)$ if and only if $C = \emptyset$.
h) $(A \cup B) \setminus B = A$ if and only if $A \cap B = \emptyset$.

Induction

Prove by induction that the following statements hold for $n \geq n_0$. Find the smallest such positive integer n_0.

1) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
2) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$
4) $\sum_{k=1}^{n} k(k+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$
5) $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$
6) $3^n > 2^n + 7n$
7) $\frac{(2n)!}{(n!)^2} < 4^{n-1}$

Inequalities

1) Prove the following inequalities.

a) $\left(1 + \frac{1}{n}\right)^n < 4,\ n \in \mathbb{N}^+$

b) $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1},\quad n \in \mathbb{N}^+$

c) $n! < \left(\frac{n+1}{2}\right)^n,\quad n \in \mathbb{N}^+$

d) $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n,\quad n \in \mathbb{N}^+, a_i > 0$

e) $\frac{x^2}{1 + x^4} \leq \frac{1}{2}$

2) Calculate the maximum value of the following functions for $x \in [0, 1]$:
a) $x^2(1 - x)$ b) $x^3 - x^5$

3) Using Bernoulli's inequality, prove that there exists a positive integer n such that

a) $0.9^n < \frac{1}{100}$ b) $\sqrt[n]{2} < 1.01$ c) $\sqrt[n]{0.1} > 0.9$

4) Prove the following inequalities.

a) $||a| - |b|| \leq |a - b|, \quad a, b \in \mathbb{R}$

b) $|a| \leq |a + b| + |b|, \quad a, b \in \mathbb{R}$